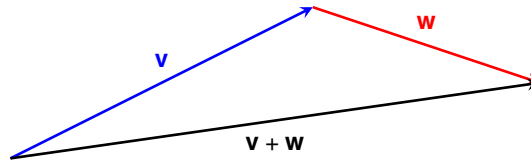

1 Triangle Inequality

The *Triangle Inequality* is a simple, yet powerful result used widely in analysis and topology as well as other branches of mathematics. The triangle inequality has its roots in geometry. It initially appeared as a proposition in the *Elements* - a treatise comprised of thirteen books covering plane and solid geometry, and number theory - written by Euclid of Alexandria around 300 B.C.

The geometric version of the triangle inequality states that the sum of the lengths of any two sides of a triangle is greater than the length the third side.

Here we are interested in the vector version of this result. Given vectors $\mathbf{v}$ and $\mathbf{w}$, we have

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$



Intuitively, we observe that equality occurs when either $\mathbf{v}$ or $\mathbf{w}$ (or both) are zero, or when non-zero vectors $\mathbf{v}$ and $\mathbf{w}$ point in the same direction, otherwise the inequality is strict.

Proving the triangle inequality requires some preliminary results.

Lemma 1. For any vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$,

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2(\mathbf{v} \cdot \mathbf{w}) + \|\mathbf{w}\|^2 \quad (1)$$

Proof By Theorem ?? we have,

$$\begin{aligned} \|\mathbf{v} + \mathbf{w}\|^2 &= (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) \\ &= \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} \\ &= \|\mathbf{v}\|^2 + 2(\mathbf{v} \cdot \mathbf{w}) + \|\mathbf{w}\|^2 \end{aligned}$$

■

Theorem 1 (Cauchy-Schwarz Inequality). For any vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$,

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\| \quad (2)$$

Proof Recall that by Theorem ??, $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$, where θ is the included angle. Our result follows from the fact that $|\cos \theta| \leq 1$. ■

Theorem 2 (Triangle Inequality). For any vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$,

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\| \quad (3)$$

Proof We will use Lemma ?? and Theorem ??.

$$\begin{aligned}\|\mathbf{v} + \mathbf{w}\|^2 &= \|\mathbf{v}\|^2 + 2(\mathbf{v} \cdot \mathbf{w}) + \|\mathbf{w}\|^2 \\ &\leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2 \\ &= (\|\mathbf{v}\| + \|\mathbf{w}\|)^2\end{aligned}$$

Taking the square root of both sides yields the desired result.

■