
1 Complex Numbers

The complex numbers, denoted $\mathbb{C}$, have the wonderful property that *every* polynomial with complex coefficients has a (complex) root. This fact is known as the *Fundamental Theorem of Algebra*.

One pleasant aspect of the complex numbers is that, whereas describing the real numbers in terms of the rationals is a rather complicated business, the complex numbers are quite easy to describe in terms of real numbers. Every *complex number* has the form

$$a + bi$$

where a and b are real numbers, and i is a root of the polynomial $x^2 + 1$. Here a and b are called the *real part* and the *imaginary part* of the complex number, respectively. The real numbers are now regarded as special complex numbers of the form $a + 0i = a$, with zero imaginary part. The complex numbers of the form $0 + bi = bi$ with zero real part are called *pure imaginary* numbers. The complex number i itself is called the *imaginary unit* and is distinguished by the fact that

$$i^2 = -1$$

As the terms *complex* and *imaginary* suggest, these numbers met with some resistance when they were first used. This has changed; now they are essential in science and engineering as well as mathematics, and they are used extensively. The names persist, however, and continue to be a bit misleading: These numbers are no more “*complex*” than the real numbers, and the number i is no more “*imaginary*” than -1 .

Much as for polynomials, two complex numbers are declared to be *equal* if and only if they have the same real parts and the same imaginary parts. In symbols,

$$a + bi = a' + b'i \quad \text{if and only if } a = a' \text{ and } b = b'$$

The addition and subtraction of complex numbers is accomplished by adding and subtracting real and imaginary parts:

$$\begin{aligned}(a + bi) + (a' + b'i) &= (a + a') + (b + b')i \\ (a + bi) - (a' + b'i) &= (a - a') + (b - b')i\end{aligned}$$

This is analogous to these operations for linear polynomials $a + bx$ and $a' + b'x$, and the multiplication of complex numbers is also analogous with one difference: $i^2 = -1$. The definition is

$$(a + bi)(a' + b'i) = (aa' - bb') + (ab' + ba')i$$

With these definitions of equality, addition, and multiplication, the complex numbers *satisfy all the basic arithmetical axioms adhered to by the real numbers* (the verifications are omitted). One consequence of this is that they can be manipulated in the obvious fashion, except that i^2 is replaced by -1 wherever it occurs, and the rule for equality must be observed.

Example 1. If $z = 2 - 3i$ and $w = -1 + i$, write each of the following in the form $a + bi$: $z + w$, $z - w$, zw , $\frac{1}{3}z$, and z^2 .

Explanation.

$$\begin{aligned}z + w &= (2 - 3i) + (-1 + i) = (2 - 1) + (-3 + 1)i = 1 - 2i \\z - w &= (2 - 3i) - (-1 + i) = (2 + 1) + (-3 - 1)i = 3 - 4i \\zw &= (2 - 3i)(-1 + i) = (-2 - 3i^2) + (2 + 3)i = 1 + 5i \\\frac{1}{3}z &= \frac{1}{3}(2 - 3i) = \frac{2}{3} - i \\z^2 &= (2 - 3i)(2 - 3i) = (4 + 9i^2) + (-6 - 6)i = -5 - 12i\end{aligned}$$

Example 2. Find all complex numbers z such as that $z^2 = i$.

Explanation. Write $z = a + bi$; we must determine a and b . Now $z^2 = (a^2 - b^2) + (2ab)i$, so the condition $z^2 = i$ becomes

$$(a^2 - b^2) + (2ab)i = 0 + i$$

Equating real and imaginary parts, we find that $a^2 = b^2$ and $2ab = 1$. The solution is $a = b = \pm \frac{1}{\sqrt{2}}$, so the complex numbers required are $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $z = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

As for real numbers, it is possible to divide by every nonzero complex number z . That is, there exists a complex number w such that $wz = 1$. As in the real case, this number w is called the *inverse* of z and is denoted by z^{-1} or $\frac{1}{z}$. Moreover, if $z = a + bi$, the fact that $z \neq 0$ means that $a \neq 0$ or $b \neq 0$. Hence $a^2 + b^2 \neq 0$, and an explicit formula for the inverse is

$$\frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

In actual calculations, the work is facilitated by two useful notions: the conjugate and the absolute value of a complex number. The next example illustrates the technique.

Example 3. Write $\frac{3 + 2i}{2 + 5i}$ in the form $a + bi$.

Explanation. Multiply top and bottom by the complex number $2 - 5i$ (obtained from the denominator by negating the imaginary part). The result is

$$\frac{3 + 2i}{2 + 5i} = \frac{(2 - 5i)(3 + 2i)}{(2 - 5i)(2 + 5i)} = \frac{(6 + 10) + (4 - 15)i}{2^2 - (5i)^2} = \frac{16}{29} - \frac{11}{29}i$$

Hence the simplified form is $\frac{16}{29} - \frac{11}{29}i$, as required.

The key to this technique is that the product $(2 - 5i)(2 + 5i) = 29$ in the denominator turned out to be a *real* number. The situation in general leads to the following notation: If $z = a + bi$ is a complex number, the *conjugate* of z is the complex number, denoted $\bar{z}$, given by

$$\bar{z} = a - bi \quad \text{where } z = a + bi$$

Hence $\bar{z}$ is obtained from z by negating the imaginary part. Thus $\overline{(2 + 3i)} = 2 - 3i$ and $\overline{(1 - i)} = 1 + i$. If we multiply $z = a + bi$ by $\bar{z}$, we obtain

$$z\bar{z} = a^2 + b^2 \quad \text{where } z = a + bi$$

The real number $a^2 + b^2$ is always nonnegative, so we can state the following definition: The *absolute value* or *modulus* of a complex number $z = a + bi$, denoted by $|z|$, is the positive square root $\sqrt{a^2 + b^2}$; that is,

$$|z| = \sqrt{a^2 + b^2} \quad \text{where } z = a + bi$$

For example, $|2 - 3i| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$ and $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Note that if a real number a is viewed as the complex number $a + 0i$, its absolute value (as a complex number) is $|a| = \sqrt{a^2}$, which agrees with its absolute value as a *real* number.

With these notions in hand, we can describe the technique applied in Example ?? as follows: When converting a quotient $\frac{z}{w}$ of complex numbers to the form $a + bi$, multiply top and bottom by the conjugate $\bar{w}$ of the denominator.

The following list contains the most important properties of conjugates and absolute values. Throughout, z and w denote complex numbers.

$$C1. \quad \overline{z \pm w} = \bar{z} \pm \bar{w}$$

$$C7. \quad \frac{1}{z} = \frac{1}{|z|^2} \bar{z}$$

$$C2. \quad \overline{zw} = \bar{z} \bar{w}$$

$$C8. \quad |z| \geq 0 \text{ for all complex numbers } z$$

$$C3. \quad \overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$$

$$C9. \quad |z| = 0 \text{ if and only if } z = 0$$

$$C4. \quad \overline{(\bar{z})} = z$$

$$C10. \quad |zw| = |z||w|$$

$$C5. \quad z \text{ is real if and only if } \bar{z} = z$$

$$C11. \quad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

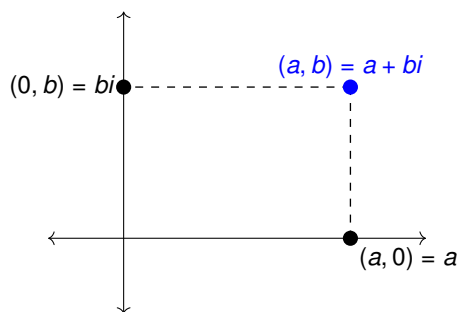
$$C6. \quad z\bar{z} = |z|^2$$

$$C12. \quad |z + w| \leq |z| + |w| \text{ (the Triangle Inequality)}$$

All these properties (except property C12) can (and should) be verified by the reader for arbitrary complex numbers $z = a + bi$ and $w = c + di$. They are not independent; for example, property C10 follows from properties C2 and C6.

The triangle inequality, as its name suggests, comes from a geometric representation of the complex numbers analogous to identification of the real numbers with the points of a line. The representation is achieved as follows:

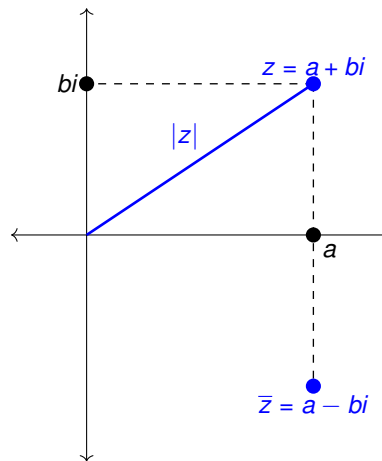
Introduce a rectangular coordinate system in the plane, and identify the complex number $a + bi$ with the point (a, b) , as shown in the figure below.



When this is done, the plane is called the *complex plane*. Note that the point $(a, 0)$ on the x axis now represents the *real* number $a = a + 0i$, and for this reason, the x axis is called the *real axis*. Similarly, the y axis is called the *imaginary axis*. The identification $(a, b) = a + bi$ of the geometric point (a, b) and the complex number $a + bi$ will be used in what follows without comment. For example, the origin will be referred to as 0.

This representation of the complex numbers in the complex plane gives a useful way of describing the absolute value and conjugate of a complex number $z = a + bi$. The absolute value $|z| = \sqrt{a^2 + b^2}$ is just the distance from z to the

origin. This makes properties C8 and C9 quite obvious. The conjugate $\bar{z} = a - bi$ of z is just the reflection of z in the real axis (x axis), a fact that makes properties C4 and C5 clear.



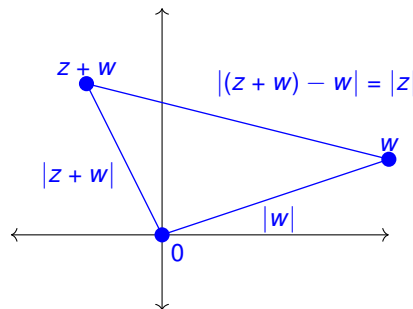
Given two complex numbers $z_1 = a_1 + b_1 i = (a_1, b_1)$ and $z_2 = a_2 + b_2 i = (a_2, b_2)$, the absolute value of their difference

$$|z_1 - z_2| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2} \quad (1)$$

is just the distance between them. This gives the *complex distance formula*:

$$|z_1 - z_2| \text{ is the distance between } z_1 \text{ and } z_2$$

This useful fact yields a simple verification of the triangle inequality, property C12. Suppose z and w are given complex numbers. Consider the triangle in the figure below whose vertices are 0 , w , and $z + w$.



The three sides have lengths $|z|$, $|w|$, and $|z + w|$ by the complex distance formula, so the inequality

$$|z + w| \leq |z| + |w|$$

expresses the obvious geometric fact that the sum of the lengths of two sides of a triangle is at least as great as the length of the third side.

The representation of complex numbers as points in the complex plane has another very useful property: It enables us to give a geometric description of the sum and product of two complex numbers. To obtain the description for the sum, let

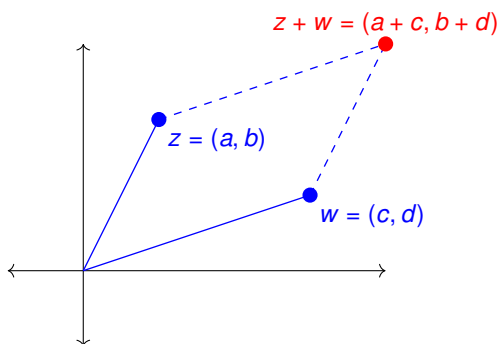
$$z = a + bi = (a, b)$$

$$w = c + di = (c, d)$$

denote two complex numbers. We claim that the four points 0, z , w , and $z + w$ form the vertices of a parallelogram. In fact, in the figure below, the lines from 0 to z and from w to $z + w$ have slopes

$$\frac{b - 0}{a - 0} = \frac{b}{a} \quad \text{and} \quad \frac{(b + d) - d}{(a + c) - c} = \frac{b}{a}$$

respectively, so these lines are parallel. (If it happens that $a = 0$, then both these lines are vertical.)



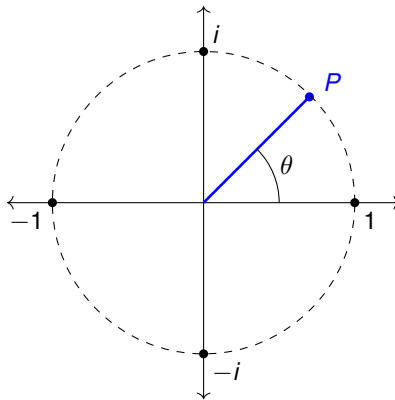
Similarly, the lines from z to $z + w$ and from 0 to w are also parallel, so the figure with vertices 0, z , w , and $z + w$ is indeed a parallelogram. Hence, the complex number $z + w$ can be obtained geometrically from z and w by *completing* the parallelogram. This is sometimes called the *parallelogram law* of complex addition. Readers who have studied mechanics will recall that velocities and accelerations add in the same way; in fact, these are all special cases of *vector* addition.

Exploration 1 In this exploration we return to the complex numbers of Example ??, and we analyze them from a geometric point of view. Let $z = 2 - 3i$ and $w = -1 + i$.

1. Plot z , w , and $z + w$ in the on the same plane to see an example of the parallelogram law of complex addition.
2. What happens geometrically when you multiply a complex number by i ? What if you multiply by i a second time? Verify that this is the same as multiplying by i^2 .

Polar Form

The geometric description of what happens when two complex numbers are multiplied is at least as elegant as the parallelogram law of addition, but it requires that the complex numbers be represented in polar form. Before discussing this, we pause to recall the general definition of the trigonometric functions sine and cosine. An angle θ in the complex plane is in *standard position* if it is measured counterclockwise from the positive real axis as indicated in the figure below.



Rather than using degrees to measure angles, it is more natural to use radian measure. This is defined as follows: The circle with its centre at the origin and radius 1 (called the *unit circle*) is drawn as in the above figure. It has circumference 2π , and the *radian measure* of θ is the length of the arc on the unit circle counterclockwise from 1 to the point P on the unit circle determined by θ . Hence $90^\circ = \frac{\pi}{2}$, $45^\circ = \frac{\pi}{4}$, $180^\circ = \pi$, and a full circle has the angle $360^\circ = 2\pi$. Angles measured clockwise from 1 are negative; for example, $-i$ corresponds to $-\frac{\pi}{2}$ (or to $\frac{3\pi}{2}$).

Consider an angle θ in the range $0 \leq \theta \leq \frac{\pi}{2}$. If θ is plotted in standard position as in the above figure, it determines a unique point P on the unit circle, and P has coordinates $(\cos \theta, \sin \theta)$ by elementary trigonometry. However, *any* angle θ (acute or not) determines a unique point on the unit circle, so we *define* the *cosine* and *sine* of θ (written $\cos \theta$ and $\sin \theta$) to be the x and y coordinates of this point. For example, the points

$$1 = (1, 0) \quad i = (0, 1) \quad -1 = (-1, 0) \quad -i = (0, -1)$$

plotted in the figure are determined by the angles $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, respectively. Hence

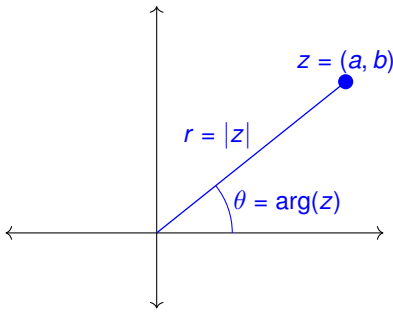
$$\begin{array}{llll} \cos 0 = 1 & \cos \frac{\pi}{2} = 0 & \cos \pi = -1 & \cos \frac{3\pi}{2} = 0 \\ \sin 0 = 0 & \sin \frac{\pi}{2} = 1 & \sin \pi = 0 & \sin \frac{3\pi}{2} = -1 \end{array}$$

Now we can describe the polar form of a complex number. Let $z = a + bi$ be a complex number, and write the absolute value of z as

$$r = |z| = \sqrt{a^2 + b^2}$$

If $z \neq 0$, the angle θ shown in the figure below is called an *argument* of z and is denoted

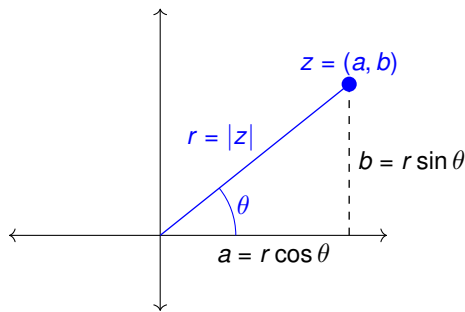
$$\theta = \arg z$$



This angle is not unique ($\theta + 2\pi k$ would do as well for any $k = 0, \pm 1, \pm 2, \dots$). However, there is only one argument θ in the range $-\pi < \theta \leq \pi$, and this is sometimes called the *principal argument* of z .

Referring to the figure below, we find that the real and imaginary parts a and b of z are related to r and θ by

$$\begin{aligned} a &= r \cos \theta \\ b &= r \sin \theta \end{aligned}$$



Hence the complex number $z = a + bi$ has the form

$$z = r(\cos \theta + i \sin \theta) \quad \text{where} \quad r = |z| \quad \text{and} \quad \theta = \arg(z)$$

The combination $\cos \theta + i \sin \theta$ is so important that a special notation is used:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

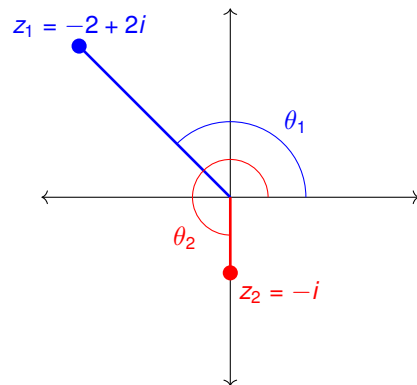
is called *Euler's formula* after the great Swiss mathematician Leonhard Euler (1707–1783). With this notation, z is written

$$z = re^{i\theta} \quad \text{where} \quad r = |z| \quad \text{and} \quad \theta = \arg(z)$$

This is a *polar form* of the complex number z . Of course it is not unique, because the argument can be changed by adding a multiple of 2π .

Example 4. Write $z_1 = -2 + 2i$ and $z_2 = -i$ in polar form.

Explanation. The two numbers are plotted in the complex plane, as shown in the figure below.



The absolute values are

$$r_1 = |-2 + 2i| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$$

$$r_2 = |-i| = \sqrt{0^2 + (-1)^2} = 1$$

By inspection, arguments of z_1 and z_2 are

$$\theta_1 = \arg(-2 + 2i) = \frac{3\pi}{4}$$

$$\theta_2 = \arg(-i) = \frac{3\pi}{2}$$

The corresponding polar forms are $z_1 = -2 + 2i = 2\sqrt{2}e^{3\pi i/4}$ and $z_2 = -i = e^{3\pi i/2}$. Of course, we could have taken the argument $-\frac{\pi}{2}$ for z_2 and obtained the polar form $z_2 = e^{-\pi i/2}$.

In Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$, the number e is the familiar constant $e = 2.71828 \dots$ from calculus. The reason for using e will not be given here; the reason why $\cos \theta + i \sin \theta$ is written as an *exponential* function of θ is that the *law of exponents* holds:

$$e^{i\theta} \cdot e^{i\phi} = e^{i(\theta+\phi)}$$

where θ and ϕ are any two angles. In fact, this is an immediate consequence of the addition identities for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$:

$$\begin{aligned}
e^{i\theta} e^{i\phi} &= (\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi) \\
&= (\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\cos \theta \sin \phi + \sin \theta \cos \phi) \\
&= \cos(\theta + \phi) + i \sin(\theta + \phi) \\
&= e^{i(\theta + \phi)}
\end{aligned}$$

This is analogous to the rule $e^a e^b = e^{a+b}$, which holds for real numbers a and b , so it is not unnatural to use the exponential notation $e^{i\theta}$ for the expression $\cos \theta + i \sin \theta$. In fact, a whole theory exists wherein functions such as e^z , $\sin z$, and $\cos z$ are studied, where z is a *complex* variable. Many deep and beautiful theorems can be proved in this theory, one of which is the so-called fundamental theorem of algebra mentioned later (Theorem ??). We shall not pursue this here.

The geometric description of the multiplication of two complex numbers follows from the law of exponents.

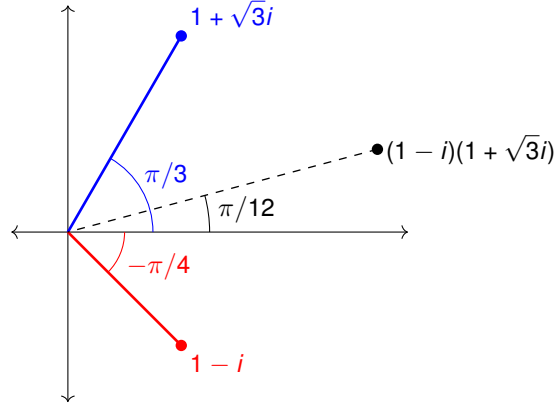
Theorem 1. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are complex numbers in polar form, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

In other words, to multiply two complex numbers, simply multiply the absolute values and add the arguments. This simplifies calculations considerably, particularly when we observe that it is valid for *any* arguments θ_1 and θ_2 .

Example 5. Multiply $(1 - i)(1 + \sqrt{3}i)$ in two ways.

Explanation. We have $|1 - i| = \sqrt{2}$ and $|1 + \sqrt{3}i| = 2$. Consider the figure below.



We have,

$$1 - i = \sqrt{2}e^{-i\pi/4}$$

$$1 + \sqrt{3}i = 2e^{i\pi/3}$$

Hence, by Theorem ??,

$$\begin{aligned}
(1 - i)(1 + \sqrt{3}i) &= (\sqrt{2}e^{-i\pi/4})(2e^{i\pi/3}) \\
&= 2\sqrt{2}e^{i(-\pi/4 + \pi/3)} \\
&= 2\sqrt{2}e^{i\pi/12}
\end{aligned}$$

This gives the required product in polar form. Of course, direct multiplication gives $(1-i)(1+\sqrt{3}i) = (\sqrt{3}+1) + (\sqrt{3}-1)i$. Hence, equating real and imaginary parts gives the formulas $\cos(\frac{\pi}{12}) = \frac{\sqrt{3}+1}{2\sqrt{2}}$ and $\sin(\frac{\pi}{12}) = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

Roots of Unity

If a complex number $z = re^{i\theta}$ is given in polar form, the powers assume a particularly simple form. In fact, $z^2 = (re^{i\theta})(re^{i\theta}) = r^2 e^{2i\theta}$, $z^3 = z^2 \cdot z = (r^2 e^{2i\theta})(re^{i\theta}) = r^3 e^{3i\theta}$, and so on. Continuing in this way, it follows by induction that the following theorem holds for any positive integer n . The name honors Abraham De Moivre (1667–1754).

Theorem 2 (De Moivre's Theorem). If θ is any angle, then $(e^{i\theta})^n = e^{in\theta}$ holds for all integers n .

Proof The case $n > 0$ has been discussed, and the reader can verify the result for $n = 0$. To derive it for $n < 0$, first observe that

$$\text{if } z = re^{i\theta} \neq 0 \text{ then } z^{-1} = \frac{1}{r} e^{-i\theta}$$

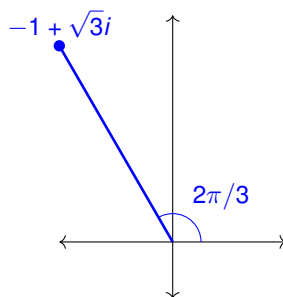
In fact, $(re^{i\theta})(\frac{1}{r} e^{-i\theta}) = 1e^{i0} = 1$ by the multiplication rule. Now assume that n is negative and write it as $n = -m$, $m > 0$. Then

$$(re^{i\theta})^n = [(re^{i\theta})^{-1}]^m = (\frac{1}{r} e^{-i\theta})^m = r^{-m} e^{i(-m\theta)} = r^n e^{in\theta}$$

If $r = 1$, this is De Moivre's theorem for negative n . ■

Example 6. Verify that $(-1 + \sqrt{3}i)^3 = 8$.

Explanation. We have $|-1 + \sqrt{3}i| = 2$, so $-1 + \sqrt{3}i = 2e^{2\pi i/3}$ (see the figure below).



De Moivre's theorem gives

$$(-1 + \sqrt{3}i)^3 = (2e^{2\pi i/3})^3 = 8e^{3(2\pi i/3)} = 8e^{2\pi i} = 8$$

De Moivre's theorem can be used to find n th roots of complex numbers where n is positive. The next example illustrates this technique.

Example 7. Find the cube roots of unity; that is, find all complex numbers z such that $z^3 = 1$.

Explanation. First write $z = re^{i\theta}$ and $1 = 1e^{i0}$ in polar form. We must use the condition $z^3 = 1$ to determine r and θ . Because $z^3 = r^3 e^{3i\theta}$ by De Moivre's theorem, this requirement becomes

$$r^3 e^{3i\theta} = 1e^{0i}$$

These two complex numbers are equal, so their absolute values must be equal and the arguments must either be equal or differ by an integral multiple of 2π :

$$\begin{aligned} r^3 &= 1 \\ 3\theta &= 0 + 2k\pi, \quad k \text{ some integer} \end{aligned}$$

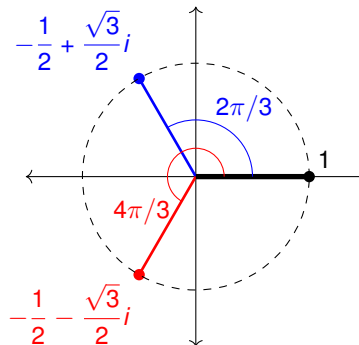
Because r is real and positive, the condition $r^3 = 1$ implies that $r = 1$. However,

$$\theta = \frac{2k\pi}{3}, \quad k \text{ some integer}$$

seems at first glance to yield infinitely many different angles for z . However, choosing $k = 0, 1, 2$ gives three possible arguments θ (where $0 \leq \theta < 2\pi$), and the corresponding roots are

$$\begin{aligned} 1e^{0i} &= 1 \\ 1e^{2\pi i/3} &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ 1e^{4\pi i/3} &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{aligned}$$

These are displayed in the figure below.



All other values of k yield values of θ that differ from one of these by a multiple of 2π —and so do not give new roots. Hence we have found all the roots.

The same type of calculation gives all complex n th roots of unity; that is, all complex numbers z such that $z^n = 1$. As before, write $1 = 1e^{0i}$ and

$$z = re^{i\theta}$$

in polar form. Then $z^n = 1$ takes the form

$$r^n e^{ni\theta} = 1e^{0i}$$

using De Moivre's theorem. Comparing absolute values and arguments yields

$$\begin{aligned} r^n &= 1 \\ n\theta &= 0 + 2k\pi, \quad k \text{ some integer} \end{aligned}$$

Hence $r = 1$, and the n values

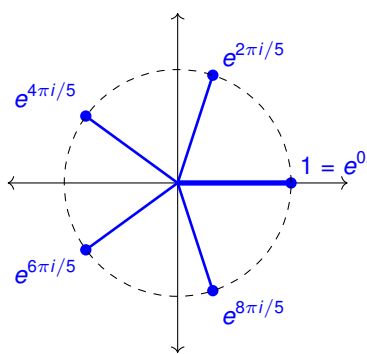
$$\theta = \frac{2k\pi}{n}, \quad k = 0, 1, 2, \dots, n-1$$

of θ all lie in the range $0 \leq \theta < 2\pi$. As in Example ??, every choice of k yields a value of θ that differs from one of these by a multiple of 2π , so these give the arguments of *all* the possible roots.

Theorem 3 (n th Roots of Unity). If $n \geq 1$ is an integer, the n th roots of unity (that is, the solutions to $z^n = 1$) are given by

$$z = e^{2\pi ki/n}, \quad k = 0, 1, 2, \dots, n-1$$

The n th roots of unity can be found geometrically as the points on the unit circle that cut the circle into n equal sectors, starting at 1. The case $n = 5$ is shown in the figure below, where the five fifth roots of unity are plotted.



The method just used to find the n th roots of unity works equally well to find the n th roots of any complex number in polar form. We give one example.

Example 8. Find the fourth roots of $\sqrt{2} + \sqrt{2}i$.

Explanation. First write $\sqrt{2} + \sqrt{2}i = 2e^{\pi i/4}$ in polar form. If $z = re^{i\theta}$ satisfies $z^4 = \sqrt{2} + \sqrt{2}i$, then De Moivre's theorem gives

$$r^4 e^{i(4\theta)} = 2e^{\pi i/4}$$

Hence $r^4 = 2$ and $4\theta = \frac{\pi}{4} + 2k\pi$, k an integer. We obtain four distinct roots (and hence all) by

$$r = \sqrt[4]{2}, \quad \theta = \frac{\pi}{16} = \frac{2k\pi}{16}, \quad k = 0, 1, 2, 3$$

Thus the four roots are

$$\sqrt[4]{2}e^{\pi i/16} \quad \sqrt[4]{2}e^{9\pi i/16} \quad \sqrt[4]{2}e^{17\pi i/16} \quad \sqrt[4]{2}e^{25\pi i/16}$$

Of course, reducing these roots to the form $a + bi$ would require the computation of $\sqrt[4]{2}$ and the sine and cosine of the various angles.

An expression of the form $ax^2 + bx + c$, where the coefficients $a \neq 0$, b , and c are real numbers, is called a *real quadratic*. A complex number u is called a *root* of the quadratic if $au^2 + bu + c = 0$. The roots are given by the famous *quadratic formula*:

$$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity $d = b^2 - 4ac$ is called the *discriminant* of the quadratic $ax^2 + bx + c$, and there is no real root if and only if $d < 0$. In this case the quadratic is said to be *irreducible*. Moreover, the fact that $d < 0$ means that $\sqrt{d} = i\sqrt{|d|}$, so the two (complex) roots are conjugates of each other:

$$u = \frac{1}{2a}(-b + i\sqrt{|d|}) \quad \text{and} \quad \bar{u} = \frac{1}{2a}(-b - i\sqrt{|d|})$$

The converse of this is true too: Given any nonreal complex number u , then u and $\bar{u}$ are the roots of some real irreducible quadratic. Indeed, the quadratic

$$x^2 - (u + \bar{u})x + u\bar{u} = (x - u)(x - \bar{u})$$

has real coefficients ($u\bar{u} = |u|^2$ and $u + \bar{u}$ is twice the real part of u) and so is irreducible because its roots u and $\bar{u}$ are not real.

Example 9. Find a real irreducible quadratic with $u = 3 - 4i$ as a root.

Explanation. We have $u + \bar{u} = 6$ and $|u|^2 = 25$, so $x^2 - 6x + 25$ is irreducible with u and $\bar{u} = 3 + 4i$ as roots.

Fundamental Theorem of Algebra

As we mentioned earlier, the complex numbers are the culmination of a long search by mathematicians to find a set of numbers large enough to contain a root of every polynomial. The fact that the complex numbers have this property was first proved by Gauss in 1797 when he was 20 years old. The proof is omitted.

Theorem 4 (Fundamental Theorem of Algebra). Every polynomial of positive degree with complex coefficients has a complex root.

If $f(x)$ is a polynomial with complex coefficients, and if u_1 is a root, then the Factor Theorem (see [Precalculus](#) by Stitz-Zeager, for instance) asserts that

$$f(x) = (x - u_1)g(x)$$

where $g(x)$ is a polynomial with complex coefficients and with degree one less than the degree of $f(x)$. Suppose that u_2 is a root of $g(x)$, again by the fundamental theorem. Then $g(x) = (x - u_2)h(x)$, so

$$f(x) = (x - u_1)(x - u_2)h(x)$$

This process continues until the last polynomial to appear is linear. Thus $f(x)$ has been expressed as a product of linear factors. The last of these factors can be written in the form $u(x - u_n)$, where u and u_n are complex (verify this), so the fundamental theorem takes the following form.

Theorem 5. Every complex polynomial $f(x)$ of degree $n \geq 1$ has the form

$$f(x) = u(x - u_1)(x - u_2) \cdots (x - u_n)$$

where $u, u_1, \dots, u_n$ are complex numbers and $u \neq 0$. The numbers $u_1, u_2, \dots, u_n$ are the roots of $f(x)$ (and need not all be distinct), and u is the coefficient of x^n .

This form of the fundamental theorem, when applied to a polynomial $f(x)$ with *real* coefficients, can be used to deduce the following result.

Theorem 6. Every polynomial $f(x)$ of positive degree with real coefficients can be factored as a product of linear and irreducible quadratic factors.

In fact, suppose $f(x)$ has the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where the coefficients a_i are real. If u is a complex root of $f(x)$, then we claim first that $\bar{u}$ is also a root. In fact, we have $f(u) = 0$, so

$$\begin{aligned} 0 = \bar{0} &= \overline{f(u)} = \overline{a_n u^n + a_{n-1} u^{n-1} + \dots + a_1 u + a_0} \\ &= \overline{a_n u^n} + \overline{a_{n-1} u^{n-1}} + \dots + \overline{a_1 u} + \overline{a_0} \\ &= \bar{a}_n \bar{u}^n + \bar{a}_{n-1} \bar{u}^{n-1} + \dots + \bar{a}_1 \bar{u} + \bar{a}_0 \\ &= a_n \bar{u}^n + a_{n-1} \bar{u}^{n-1} + \dots + a_1 \bar{u} + a_0 \\ &= f(\bar{u}) \end{aligned}$$

where $\bar{a}_i = a_i$ for each i because the coefficients a_i are real. Thus if u is a root of $f(x)$, so is its conjugate $\bar{u}$. Of course some of the roots of $f(x)$ may be real (and so equal their conjugates), but the nonreal roots come in pairs, u and $\bar{u}$. By Theorem ??, we can thus write $f(x)$ as a product:

$$f(x) = a_n(x - r_1) \cdots (x - r_k)(x - u_1)(x - \bar{u}_1) \cdots (x - u_m)(x - \bar{u}_m) \quad (2)$$

where a_n is the coefficient of x^n in $f(x)$; $r_1, r_2, \dots, r_k$ are the real roots; and $u_1, \bar{u}_1, u_2, \bar{u}_2, \dots, u_m, \bar{u}_m$ are the nonreal roots. But the product

$$(x - u_j)(x - \bar{u}_j) = x^2 - (u_j + \bar{u}_j)x + (u_j \bar{u}_j)$$

is a real irreducible quadratic for each j (see the discussion preceding Example ??). Hence (??) shows that $f(x)$ is a product of linear and irreducible quadratic factors, each with real coefficients. This is the conclusion in Theorem ??.

Practice Problems

Problem 2 Solve each of the following for the real number x .

1. $x - 4i = (2 - i)^2$

$$x = \dots$$

2. $(2 + xi)(3 - 2i) = 12 + 5i$

3. $(2 + xi)^2 = 4$

$$x = \dots \text{ or } x = \dots$$

4. $(2 + xi)(2 - xi) = 5$

Problem 3 Convert each of the following to the form $a + bi$.

1. (challenge problem) $(2 - 3i) - 2(2 - 3i) + 9$

2. (challenge problem) $(3 - 2i)(1 + i) + |3 + 4i|$

$$\dots + i$$

$$3. \frac{1+i}{2-3i} + \frac{1-i}{-2+3i}$$

$$4. \frac{3-2i}{1-i} + \frac{3-7i}{2-3i}$$

$$\frac{\dots\dots}{\dots\dots} + \frac{\dots\dots}{\dots\dots}i$$

$$5. i^{131}$$

$$6. (2-i)^3$$

$$\dots\dots + \dots\dots i$$

$$7. (1+i)^4$$

$$\dots\dots + \dots\dots i$$

$$8. (1-i)^2(2+i)^2$$

$$9. \text{ (challenge problem) } \frac{3\sqrt{3}-i}{\sqrt{3}+i} + \frac{\sqrt{3}+7i}{\sqrt{3}-i}$$

Problem 4 In each case, find the complex number z .

$$1. iz - (1+i)^2 = 3-i$$

$$2. \text{ (challenge problem) } (i+z) - 3i(2-z) = iz+1$$

$$\frac{\dots\dots}{\dots\dots} + \frac{\dots\dots}{\dots\dots}i$$

$$3. z^2 = -i$$

$$4. z^2 = 3-4i$$

$$\pm(\dots\dots - i)$$

$$5. z(1+i) = \bar{z} + (3+2i)$$

$$6. \text{ (challenge problem) } z(2-i) = (\bar{z}+1)(1+i)$$

Click the arrow to see the answer.

$$1+i$$

Problem 5 In each case, find the roots of the real quadratic equation.

$$1. x^2 - 2x + 3 = 0$$

$$2. x^2 - x + 1 = 0$$

Click the arrow to see the answer.

$$\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$3. 3x^2 - 4x + 2 = 0$$

4. $2x^2 - 5x + 2 = 0$

Click the arrow to see the answer.

$$2, \frac{1}{2}$$

Problem 6 Find all numbers x in each case.

1. $x^3 = 8$

2. $x^3 = -8$

Click the arrow to see the answer.

$$-2, 1 \pm \sqrt{3}i$$

3. $x^4 = 16$

4. $x^4 = 64$

Click the arrow to see the answer.

$$\pm 2\sqrt{2}, \pm 2\sqrt{i}$$

Problem 7 In each case, find a real quadratic with u as a root, and find the other root.

1. $u = 1 + i$

2. $u = 2 - 3i$

Click the arrow to see the answer.

$$x^2 - 4x + 13; 2 + 3i$$

3. $u = -i$

4. $u = 3 - 4i$

Click the arrow to see the answer.

$$x^2 - 6x + 25; 3 + 4i$$

Problem 8 Find the roots of $x^2 - 2 \cos \theta x + 1 = 0$, θ any angle.

Problem 9 Find a real polynomial of degree 4 with $2 - i$ and $3 - 2i$ as roots.

Click the arrow to see the answer.

$$x^4 - 10x^3 + 42x^2 - 82x + 65$$

Problem 10 Let $\operatorname{re} z$ and $\operatorname{im} z$ denote, respectively, the real and imaginary parts of z . Show that:

1. $\operatorname{im}(iz) = \operatorname{re} z$

2. $\operatorname{re}(iz) = -\operatorname{im} z$

3. $z + \bar{z} = 2\operatorname{re} z$

4. $z - \bar{z} = 2i\operatorname{im} z$

-
5. (challenge problem) $\operatorname{re}(z + w) = \operatorname{re} z + \operatorname{re} w$, and $\operatorname{re}(tz) = t \cdot \operatorname{re} z$ if t is real
 6. (challenge problem) $\operatorname{im}(z + w) = \operatorname{im} z + \operatorname{im} w$, and $\operatorname{im}(tz) = t \cdot \operatorname{im} z$ if t is real

Problem 11 In each case, show that u is a root of the quadratic equation, and find the other root.

1. (challenge problem) $x^2 - 3ix + (-3 + i) = 0$; $u = 1 + i$
2. (challenge problem) $x^2 + ix - (4 - 2i) = 0$; $u = -2$
3. (challenge problem) $x^2 - (3 - 2i)x + (5 - i) = 0$; $u = 2 - 3i$
4. (challenge problem) $x^2 + 3(1 - i)x - 5i = 0$; $u = -2 + i$

Problem 12 Find the roots of each of the following complex quadratic equations.

1. $x^2 + 2x + (1 + i) = 0$
2. $x^2 - x + (1 - i) = 0$

Click the arrow to see the answer.

$$-i, 1 + i$$

3. (challenge problem) $x^2 - (2 - i)x + (3 - i) = 0$
4. (challenge problem) $x^2 - 3(1 - i)x - 5i = 0$

Click the arrow to see the answer.

$$2 - i, 1 - 2i$$

Problem 13 In each case, describe the graph of the equation (where z denotes a complex number).

1. $|z| = 1$
2. $|z - 1| = 2$

Click the arrow to see the answer. Circle, centre at 1, radius 2

3. $z = i\bar{z}$
4. $z = -\bar{z}$

Click the arrow to see the answer. Imaginary axis

5. $z = |z|$
6. $\operatorname{im} z = m \cdot \operatorname{re} z$, m a real number

Click the arrow to see the answer. Line $y = mx$

Problem 14 a. Verify $|zw| = |z||w|$ directly for $z = a + bi$ and $w = c + di$.

b. Deduce (a) from properties C2 and C6.

Problem 15 Prove that $|z + w| = |z|^2 + |w|^2 + w\bar{z} + \bar{w}z$ for all complex numbers w and z .

Problem 16 If zw is real and $z \neq 0$, show that $w = a\bar{z}$ for some real number a .

Problem 17 If $zw = \bar{z}v$ and $z \neq 0$, show that $w = uv$ for some u in $\mathbb{C}$ with $|u| = 1$.

Problem 18 Show that $(1 + i)^n + (1 - i)^n$ is real for all n , using property C5.

Problem 19 Express each of the following in polar form (use the principal argument).

1. $3 - 3i$

2. $-4i$

Click the arrow to see the answer.

$$4e^{-\pi i/2}$$

3. $-\sqrt{3} + i$

4. $-4 + 4\sqrt{3}i$

Click the arrow to see the answer.

$$8e^{2\pi i/3}$$

5. $-7i$

6. $-6 + 6i$

Click the arrow to see the answer.

$$6\sqrt{2}e^{3\pi i/4}$$

Problem 20 Express each of the following in the form $a + bi$.

1. $3e^{\pi i}$

2. $e^{7\pi i/3}$

Click the arrow to see the answer.

$$\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

3. $2e^{3\pi i/4}$

4. $\sqrt{2}e^{-\pi i/4}$

Click the arrow to see the answer.

$$1 - i$$

5. $e^{5\pi i/4}$

6. $2\sqrt{3}e^{-2\pi i/6}$

Click the arrow to see the answer.

$$\sqrt{3} - 3i$$

Problem 21 Express each of the following in the form $a + bi$.

1. $(-1 + \sqrt{3}i)^2$

2. $(1 + \sqrt{3}i)^{-4}$

Click the arrow to see the answer.

$$-\frac{1}{32} + \frac{\sqrt{3}}{32}i$$

3. $(1 + i)^8$

4. $(1 - i)^{10}$

Click the arrow to see the answer.

$$-32i$$

5. $(1 - i)^6(\sqrt{3} + i)^3$

6. $(\sqrt{3} - i)^9(2 - 2i)^5$

Click the arrow to see the answer.

$$-2^{16}(1 + i)$$

Problem 22 Use De Moivre's theorem to show that:

a. $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$; $\sin 2\theta = 2 \cos \theta \sin \theta$

b. $\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$;
 $\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$

Problem 23 a. Find the fourth roots of unity.

b. Find the sixth roots of unity.

Problem 24 Find all complex numbers z such that:

1. $z^4 = -1$

2. $z^4 = 2(\sqrt{3}i - 1)$

Click the arrow to see the answer.

$$\pm \frac{\sqrt{2}}{2}(\sqrt{3} + i), \pm \frac{\sqrt{2}}{2}(-1 + \sqrt{3}i)$$

3. $z^3 = -27i$

4. $z^6 = -64$

Click the arrow to see the answer.

$$\pm 2i, \pm(\sqrt{3} + i), \pm(\sqrt{3} - i)$$

Problem 25 If $z = re^{i\theta}$ in polar form, show that:

-
1. $\bar{z} = re^{-i\theta}$
 2. $z^{-1} = \frac{1}{r}e^{-i\theta}$ if $z \neq 0$

Problem 26 Show that the sum of the n th roots of unity is zero.

Problem 27 1. Let $z_1, z_2, z_3, z_4,$ and z_5 be equally spaced around the unit circle. Show that $z_1 + z_2 + z_3 + z_4 + z_5 = 0$.

2. Repeat (a) for any $n \geq 2$ points equally spaced around the unit circle.

3. If $|w| = 1$, show that the sum of the roots of $z^n = w$ is zero.

Problem 28 If z^n is real, $n \geq 1$, show that $(\bar{z})^n$ is real.

Problem 29 If $\bar{z}^2 = z^2$, show that z is real or pure imaginary.

Problem 30 If a and b are rational numbers, let p and q denote numbers of the form $a + b\sqrt{2}$. If $p = a + b\sqrt{2}$, define $\bar{p} = a - b\sqrt{2}$ and $[p] = a^2 - 2b^2$. Show that each of the following holds.

1. (challenge problem) $a + b\sqrt{2} = a_1 + b_1\sqrt{2}$ only if $a = a_1$ and $b = b_1$
2. $\widetilde{p \pm q} = \bar{p} \pm \bar{q}$
3. $\widetilde{pq} = \bar{p}\bar{q}$
4. $[p] = p\bar{p}$
5. $[pq] = [p][q]$
6. (challenge problem) If $f(x)$ is a polynomial with rational coefficients and $p = a + b\sqrt{2}$ is a root of $f(x)$, then $\bar{p}$ is also a root of $f(x)$.

Text Source

This section was adapted from Appendix A of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 581–594.