
1 Challenge Problems

Challenge Problems for Chapter 10

Problem 1 Given linear transformations

$$\begin{array}{ccccc} \mathbb{R}^2 & \xrightarrow{T_1} & \mathbb{R}^2 & \xrightarrow{T_2} & \mathbb{R}^2 \\ & \searrow & & \nearrow & \\ & & T & & \end{array}$$

1. If T_1 and T_2 are both one-to-one, show that T is one-to-one.
2. If T_1 and T_2 are both onto, show that T is onto.

Problem 2 Let $T : V \rightarrow W$ be a linear transformation.

1. If T is one-to-one and $TR_1 = TR_2$ for linear transformations $R_1, R_2 : U \rightarrow V$, show that $R_1 = R_2$.
2. If T is onto and $S_1T = S_2T$ for linear transformations $S_1, S_2 : W \rightarrow U$, show that $S_1 = S_2$.

Problem 3 Consider functions defined on $\{1, 2, \dots, n\}$ having values in $\mathbb{R}$. Explain how, if V is the set of all such functions, V can be considered as $\mathbb{R}^n$.

Problem 4 Let $T : V \rightarrow U$ and $S : U \rightarrow W$ be linear transformations.

1. If ST is one-to-one, show that T is one-to-one and that $\dim V \leq \dim U$.
2. If ST is onto, show that S is onto and that $\dim W \leq \dim U$.

Problem 5 Let $\mathbb{D}_n$ denote the space of all functions $f : \{1, 2, \dots, n\} \rightarrow \mathbb{R}^n$ (see Problem ??). If $T : \mathbb{D}_n \rightarrow \mathbb{R}^n$ is defined by

$$T(f) = (f(1), f(2), \dots, f(n))$$

show that T is an isomorphism.

Problem 6 Let $\mathbb{R}^{\mathbb{N}}$ denote the set of all real valued sequences. For $\mathbf{a} \equiv \{a_n\}_{n=1}^{\infty}$, $\mathbf{b} \equiv \{b_n\}_{n=1}^{\infty}$ two of these, define their sum to be given by

$$\mathbf{a} + \mathbf{b} = \{a_n + b_n\}_{n=1}^{\infty}$$

and define scalar multiplication by

$$c\mathbf{a} = \{ca_n\}_{n=1}^{\infty} \text{ where } \mathbf{a} = \{a_n\}_{n=1}^{\infty}$$

Is this a vector space?

Problem 7 Let $\mathbb{C}^2$ be the set of ordered pairs of complex numbers. Define addition and scalar multiplication in the usual way.

$$(z, w) + (\hat{z}, \hat{w}) = (z + \hat{z}, w + \hat{w}), \quad u(z, w) \equiv (uz, uw)$$

Here the scalars are from $\mathbb{C}$. Show this is a vector space.

Bibliography

Several problems come from Chapter 9 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 469–535.

Several problems come from the end of Chapter 7 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 396.