
1 Challenge Problems

Challenge Problems for Chapter 8

Problem 1 Suppose A is an $n \times n$ matrix consisting entirely of real entries but $a + ib$ is a complex eigenvalue having the eigenvector, $\mathbf{x} + i\mathbf{y}$. Here $\mathbf{x}$ and $\mathbf{y}$ are real vectors. Show that then $a - ib$ is also an eigenvalue with the eigenvector, $\mathbf{x} - i\mathbf{y}$.

Problem 2 If A is 2×2 and diagonalizable, show that $C(A) = \{X \mid XA = AX\}$ has dimension 2 or 4.

Problem 3 Let A be $n \times n$ with n distinct real eigenvalues. If $AC = CA$, show that C is diagonalizable.

Problem 4 Given a polynomial $p(z) = r_0 + r_1z + \cdots + r_nz^n$ and a square matrix A , the matrix $p(A) = r_0I + r_1A + \cdots + r_nA^n$ is called the evaluation of $p(z)$ at A . Let $B = P^{-1}AP$. Show that $p(B) = P^{-1}p(A)P$ for all polynomials $p(x)$.

Problem 5 If A is diagonalizable and $p(x)$ is a polynomial such that $p(\lambda) = 0$ for all eigenvalues λ of A , show that $p(A) = O$ (here, the final O is the zero matrix the same size as A).

Remark 1. The characteristic polynomial of A (see Definition ??) certainly satisfies the requirement that $p(\lambda) = 0$ for all eigenvalues λ of A . In solving this problem you have proved a special case of the Cayley-Hamilton theorem, see Theorem ?. In fact, if $p(\lambda)$ is the characteristic polynomial of A , then $p(A) = 0$ whether or not A is diagonalizable.

Problem 6 Let P be an invertible $n \times n$ matrix. If A is any $n \times n$ matrix, write $T_P(A) = P^{-1}AP$. Verify that:

1. $T_P(I) = I$
2. $T_P(AB) = T_P(A)T_P(B)$
3. $T_P(A + B) = T_P(A) + T_P(B)$
4. $T_P(rA) = rT_P(A)$
5. $T_P(A^k) = [T_P(A)]^k$ for $k \geq 1$
6. If A is invertible, $T_P(A^{-1}) = [T_P(A)]^{-1}$.
7. If Q is invertible, $T_Q[T_P(A)] = T_{PQ}(A)$.

Problem 7 Let $A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$ and $B = \begin{bmatrix} c & a & b \\ a & b & c \\ b & c & a \end{bmatrix}$.

1. Show that $x^3 - (a^2 + b^2 + c^2)x - 2abc$ has real roots by considering A .
2. Show that $a^2 + b^2 + c^2 \geq ab + ac + bc$ by considering B .

Problem 8 Assume the 2×2 matrix A is similar to an upper triangular matrix. If $\text{tr}A = 0 = \text{tr}A^2$, show that A^2 is equal to the zero matrix.

Problem 9 Show that A is similar to A^T for all 2×2 matrices A .

Problem 10 Suppose A is an $n \times n$ matrix and let $\mathbf{v}$ be an eigenvector such that $A\mathbf{v} = \lambda\mathbf{v}$. Also suppose the characteristic polynomial of A is

$$\det(zI - A) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$$

Explain why

$$(A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I)\mathbf{v} = 0$$

Use this to prove that the Cayley-Hamilton theorem holds for any diagonalizable matrix A . (The Cayley-Hamilton theorem says that A satisfies its characteristic equation, i.e.,

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I = 0$$

. (For a proof of the general case, see Theorem ??)

Problem 11 Suppose the characteristic polynomial of an $n \times n$ matrix A is $\lambda^n - 1$. Find A^{nm} where m is an integer. Click the arrow to see answer. The eigenvalues are distinct because they are the n^{th} roots of 1. Hence if $\mathbf{x}$ is a given vector with

$$\mathbf{x} = \sum_{j=1}^n a_j \mathbf{v}_j$$

then

$$A^{nm}\mathbf{x} = A^{nm} \sum_{j=1}^n a_j \mathbf{v}_j = \sum_{j=1}^n a_j A^{nm} \mathbf{v}_j = \sum_{j=1}^n a_j \mathbf{v}_j = \mathbf{x}$$

so $A^{nm} = I$.

Bibliography

These problems come from Chapter 7 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

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