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# 1 Challenge Problems

## Challenge Problems for Chapter 4

**Problem 1** Solve for the matrix  $X$  if:

1.  $PXQ = R$ ;
2.  $XP = S$ ;

$$\text{where } P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix},$$
$$R = \begin{bmatrix} -1 & 1 & -4 \\ -4 & 0 & -6 \\ 6 & 6 & -6 \end{bmatrix}, S = \begin{bmatrix} 1 & 6 \\ 3 & 1 \end{bmatrix}$$

**Problem 2** Consider

$$p(X) = X^3 - 5X^2 + 11X - 4I.$$

1. If  $p(U) = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$  compute  $p(U^T)$ .
2. If  $p(U) = 0$  where  $U$  is  $n \times n$ , find  $U^{-1}$  in terms of  $U$ .

Click on the arrow to see answer.  $U^{-1} = \frac{1}{4}(U^2 - 5U + 11I)$ .

**Problem 3** Assume that a system  $A\mathbf{x} = \mathbf{b}$  of linear equations has at least two distinct solutions  $\mathbf{y}$  and  $\mathbf{z}$ .

1. Show that  $\mathbf{x}_k = \mathbf{y} + k(\mathbf{y} - \mathbf{z})$  is a solution for every  $k$ .
2. Show that  $\mathbf{x}_k = \mathbf{x}_m$  implies  $k = m$ .
3. Deduce that  $A\mathbf{x} = \mathbf{b}$  has infinitely many solutions.

Click on the arrow to see answer.

1. If  $\mathbf{x}_k = \mathbf{x}_m$ , then  $\mathbf{y} + k(\mathbf{y} - \mathbf{z}) = \mathbf{y} + m(\mathbf{y} - \mathbf{z})$ . So  $(k - m)(\mathbf{y} - \mathbf{z}) = \mathbf{0}$ . But  $\mathbf{y} - \mathbf{z}$  is not zero (because  $\mathbf{y}$  and  $\mathbf{z}$  are distinct), so  $k - m = 0$ .

**Problem 4** Let  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

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1. show that  $A^2 = I$ .

2. What is wrong with the following argument? If  $A^2 = I$ , then  $A^2 - I = 0$ , so  $(A - I)(A + I) = 0$ , whence  $A = I$  or  $A = -I$ .

**Problem 5** Let  $E$  and  $F$  be elementary matrices obtained from the identity matrix by adding multiples of row  $k$  to rows  $p$  and  $q$ . If  $k \neq p$  and  $k \neq q$ , show that  $EF = FE$ .

## Bibliography

These exercises come from the end of Chapter 2 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 143–144.