
1 Challenge Problems

Challenge Problems for Chapter 9

Problem 1 If U is a subspace of $\mathbb{R}^n$, show that $(U^\perp)^\perp = U$.

Problem 2 If W is a subspace of $\mathbb{R}^n$, show how to find an $n \times n$ matrix A such that $W = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$.

Problem 3 If U and W are subspaces, define $U + W$ to be the set of all possible sums of elements of U with elements of W .

1. Is $U + W$ a subspace? Show that $(U + W)^\perp = U^\perp \cap W^\perp$.
2. Illustrate this result with a diagram in $\mathbb{R}^3$ where U and W are two distinct lines through the origin. What does $U + W$ look like? What do $U^\perp$ and $W^\perp$ look like?

Problem 4 A square matrix E satisfying $E^2 = E = E^T$ is called a projection matrix. If E is a projection matrix, show that $I - E$ is also a projection matrix.

Problem 5 Let E be an $n \times n$ matrix, and let $W = \{\mathbf{x}E \mid \mathbf{x} \text{ in } \mathbb{R}^n\}$. Show that the following are equivalent.

1. $E^2 = E = E^T$ (E is a projection matrix).
2. $(\mathbf{x} - \mathbf{x}E) \cdot (\mathbf{y}E) = 0$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.
3. $\text{proj}_W(\mathbf{x}) = \mathbf{x}E$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Problem 6 If $EF = 0 = FE$ and E and F are projection matrices, show that $E + F$ is also a projection matrix.

Problem 7 If A is $m \times n$ and AA^T is invertible, show that $E = A^T(AA^T)^{-1}A$ is a projection matrix.

Click the arrow to see answer.

$$E^T = A^T[(AA^T)^{-1}]^T(A^T)^T = A^T[(AA^T)^T]^{-1}A = A^T[AA^T]^{-1}A = E$$

Problem 8 Consider $A = \begin{bmatrix} 0 & a & 0 \\ a & 0 & c \\ 0 & c & 0 \end{bmatrix}$ where one of $a, c \neq 0$. Show that the characteristic polynomial (see Definition ??) is given by $c_A(z) = z(z - k)(z + k)$, where $k = \sqrt{a^2 + c^2}$, and find an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal.

Click the arrow to see the answer. $Q = \frac{1}{\sqrt{2}k} \begin{bmatrix} c\sqrt{2} & a & a \\ 0 & k & -k \\ -a\sqrt{2} & c & c \end{bmatrix}$

Problem 9 Given $A = \begin{bmatrix} b & a \\ a & b \end{bmatrix}$, show that the characteristic polynomial (see Definition ??) is given by $c_A(z) = (z - a - b)(z + a - b)$, and find an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal.

Bibliography

Some of these problems come from Section 7.4 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 433–438.

Other problems come from the second part of Section 8.1 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p. 422–423