
1 Challenge Problems

Challenge Problems for Chapter 5

Problem 1 Suppose V has dimension p and W has dimension q and they are each contained in a subspace, U which has dimension equal to n where $n > \max(p, q)$. What are the possibilities for the dimension of $V \cap W$?

Click the arrow to see the answer. Let $\{x_1, \dots, x_k\}$ be a basis for $V \cap W$. Then there is a basis for V and W which are respectively

$$\{x_1, \dots, x_k, y_{k+1}, \dots, y_p\}, \{x_1, \dots, x_k, z_{k+1}, \dots, z_q\}$$

It follows that you must have $k + p - k + q - k \leq n$ and so you must have

$$p + q - n \leq k$$

Problem 2 Let $\{v_1, v_2, \dots, v_n\}$ be a basis of $\mathbb{R}^n$. Let A be an $n \times n$ matrix.

1. If A is invertible, show that $\{Av_1, Av_2, \dots, Av_n\}$ is a basis of $\mathbb{R}^n$.
2. If $\{Av_1, Av_2, \dots, Av_n\}$ is a basis of $\mathbb{R}^n$, show that A is invertible.

Problem 3 Show that $\text{null}(A) = \text{null}(A^T A)$ for any real matrix A .

Problem 4 Let A be an $m \times n$ matrix of rank r . Show that $\dim(\text{null}(A)) = n - r$.

Problem 5 Let U and W be subspaces of $\mathbb{R}^3$. If $\dim(U) = \dim(W) = 2$, and $U \neq W$, show that $\dim(U \cap W) = 1$.

Bibliography

These problems came from Chapter 6 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)
W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 358–359, and p. 370.