
1 Properties of the Determinant

We begin by summarizing the properties of determinants we introduced in previous sections.

Summary 1. • The determinant of the identity matrix is equal to 1. (Lemma ??)

- The determinant of a triangular matrix is equal to the product of the main diagonal entries. (Lemma ??)
- The determinant of the transpose is equal to the determinant of the matrix. (Theorem ??)
- If a matrix contains a row of zeros, then its determinant is equal to 0. (Lemma ??)
- If two rows of a matrix are the same, then the determinant of the matrix is equal to 0. (Lemma ??)
- If one row of a matrix is a scalar multiple of another row, then the determinant of the matrix is equal to 0. (Lemma ??)
- If B is obtained from A by interchanging two different rows, then

$$\det B = -\det A$$

- If B is obtained from A by multiplying one of the rows of A by a non-zero constant k . Then

$$\det B = k \det A$$

- If B is obtained from A by adding a multiple of one row of A to another row, then

$$\det B = \det A$$

(The last three properties comprise Theorem ??)

In this section we will prove the following important results:

1. A square matrix is singular if and only if its determinant is equal to 0.
2. The determinant of a product is the product of the determinants.

To get us started, we need the following lemma.

Lemma 1. Let A be a square matrix, and let E be an elementary matrix, then

$$\det EA = \det E \det A$$

Proof Recall that if E is obtained from I using an elementary row operation, then the same elementary row operation carries A to EA . There are three types of elementary row operations and three types of elementary matrices, so we will have to consider three cases.

Case 1. Suppose E is obtained from I by interchanging two rows, then

$$\det E = -1 \quad \text{and} \quad \det EA = -\det A$$

so

$$\det EA = \det E \det A$$

Case 2. Suppose E is obtained from I by multiplying one of the rows of I by a non-zero constant k , then

$$\det E = k \quad \text{and} \quad \det EA = k \det A$$

so

$$\det EA = \det E \det A$$

Case 3. Suppose E is obtained from I by adding a scalar multiple of one row to another row, then

$$\det E = 1 \quad \text{and} \quad \det EA = \det A$$

so

$$\det EA = \det E \det A$$

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Invertibility and the Determinant

Recall that we first introduced determinants in the context of invertibility of 2×2 matrices. Specifically, we found that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $\det A \neq 0$. (A logically equivalent statement is: A is singular if and only if $\det A = 0$.) We are now in the position to prove this result for all square matrices.

Theorem 1. A square matrix A is singular if and only if $\det A = 0$.

Proof Let A be a square matrix. To determine whether A is singular we need to find $\text{rref}(A)$. In [Elementary Matrices](#) we found that there exist elementary matrices $E_1, \dots, E_k$ such that

$$E_k \dots E_2 E_1 A = \text{rref}(A)$$

so

$$\det(E_k \dots E_2 E_1 A) = \det(\text{rref}(A))$$

By repeated application of Lemma ??, we find that

$$\det E_k \dots \det E_2 \det E_1 \det A = \det(\text{rref}(A))$$

Suppose that A is singular, then $\text{rref}(A) \neq I$. But then $\text{rref}(A)$ contains a row of zeros, and $\det(\text{rref}(A)) = 0$. (Lemma ??) Since determinants of elementary matrices are non-zero, we conclude that $\det A = 0$.

Conversely, suppose $\det A = 0$, then

$$\det(\text{rref}(A)) = \det E_k \dots \det E_2 \det E_1 \det A = 0$$

But then $\text{rref}(A) \neq I$, so A is singular.

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Example 1. Determine whether A is an invertible matrix without using elementary row operations.

$$A = \begin{bmatrix} 3 & 4 & -8 \\ 1 & 8 & -10 \\ 1 & -2 & 1 \end{bmatrix}$$

Explanation. Compute the determinant of A . You will find that $\det A = 0$. By Theorem ?? we conclude that A is not invertible.

Determinant of a Product

Theorem 2. Let A and B be square matrices, then

$$\det AB = \det A \det B$$

Proof Suppose A is invertible, then A can be written as a product of elementary matrices. (Theorem ??)

$$A = E_1 E_2 \dots E_k$$

Then, by repeated application of Lemma ??, we get

$$\begin{aligned}\det AB &= \det (E_1 E_2 \dots E_k B) \\ &= \det E_1 \det E_2 \dots \det E_k \det B \\ &= \det (E_1 E_2 \dots E_k) \det B \\ &= \det A \det B\end{aligned}$$

Now suppose that A is not invertible. Then AB is also not invertible. So, $\det A = 0$ and $\det AB = 0$. Thus $\det AB = 0 = \det A \det B$. ■

The following theorem is a nice consequence of Theorem ??. We leave the proof to the reader. (Practice Problem ??)

Theorem 3. Let A be a nonsingular matrix, then

$$\det A^{-1} = \frac{1}{\det A}$$

Practice Problems

Problem 1 Without doing written computations, determine whether matrix A is singular.

$$A = \begin{bmatrix} 0 & 0 & -3 \\ 2 & 3 & -1 \\ 0 & 0 & 10 \end{bmatrix}$$

Multiple Choice:

1. A is singular
2. A is nonsingular

Problem 2 Without doing written computations, determine whether matrix A is singular.

$$A = \begin{bmatrix} 2 & -3 & -1 \\ 0 & 4 & 5 \\ 0 & 0 & -3 \end{bmatrix}$$

Multiple Choice:

1. A is singular
2. A is nonsingular

Problem 3 Show that all matrices of the form

$$\begin{bmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{bmatrix}$$

are singular.

Problem 4 Find values of x for which the given matrix is singular.

$$\begin{bmatrix} 2-x & 1 \\ 5 & 6-x \end{bmatrix}$$

List values of x in an increasing order.

Answer:

$$x = \dots \dots \quad x = \dots \dots$$

Problem 5 Suppose A and B are 2×2 matrices such that $\det A = 2$ and $\det B = \frac{1}{3}$. Then, $\det AB^{-1} = \dots \dots$

Problem 6 Suppose A and B are 2×2 matrices such that $\det A = 2$ and $\det B = \frac{1}{3}$. Then, $\det (AB)^{-1} = \dots \dots$

Problem 7 Suppose A and B are 2×2 matrices such that $\det A = 2$ and $\det B = \frac{1}{3}$. Then, $\det 2AB = \dots \dots$

Problem 8 Prove or give a counterexample.

$$\det(A+B) = \det(A^T + B^T)$$

Problem 9 Prove that for a nonsingular matrix A

$$\det A^{-1} = \frac{1}{\det A}$$

(Theorem ??)

Problem 10 Suppose A is an invertible matrix such that

$$A^{-1} = A^T$$

What are the possible values for $\det A$? List your answers in increasing order.

Answer: $\det A = \dots \dots, \dots \dots$