
1 Tedious Proofs Concerning Determinants

In [Finding the Determinant](#) we described the determinant as a function that assigns a scalar to every square matrix. The value of the function in the original definition was given by cofactor expansion along the first row of the matrix. We also observed, through examples, that cofactor expansion along any row or column produces the same value. Examples, however, do not constitute a sufficient proof of equivalency of different cofactor expansions. In this section we will prove that cofactor expansions along any row or column produce the same outcome. This result is known as the Laplace Expansion Theorem. We will also prove several results concerning elementary row operations.

Cofactor Expansion Along the Top Row

Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Definition ?? of cofactor expansion along the top row for a 3×3 matrix requires three *minor* matrices associated with A .

- A_{11} is obtained from A by deleting the first row and the first column of A .

$$\begin{bmatrix} a & b & c \\ d & \boxed{e} & f \\ g & h & i \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} e & f \\ h & i \end{bmatrix}$$

- A_{12} is obtained from A by deleting the first row and the second column of A .

$$\begin{bmatrix} a & b & c \\ \boxed{d} & e & f \\ g & h & \boxed{i} \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} d & f \\ g & i \end{bmatrix}$$

- A_{13} is obtained from A by deleting the first row and the third column of A .

$$\begin{bmatrix} a & b & c \\ \boxed{d} & \boxed{e} & f \\ g & h & i \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

The determinant of A is given by

$$\begin{aligned} \det A = |A| &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(\det A_{11}) - b(\det A_{12}) + c(\det A_{13}) \end{aligned}$$

Now we are ready for an $n \times n$ matrix. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j-1} & a_{1j} & a_{1j+1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j-1} & a_{2j} & a_{2j+1} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj-1} & a_{nj} & a_{nj+1} & \dots & a_{nn} \end{bmatrix}$$

Define A_{1j} to be an $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column of A . We say that A_{1j} is the $(1, j)$ -minor of A .

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j-1} & a_{1j} & a_{1j+1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j-1} & a_{2j} & a_{2j+1} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj-1} & a_{nj} & a_{nj+1} & \dots & a_{nn} \end{bmatrix}$$

For a 3×3 matrix A we have

$$\det A = a_{11}(\det A_{11}) - a_{12}(\det A_{12}) + a_{13}(\det A_{13})$$

We want to follow the same pattern to define the determinant of a larger matrix. A distinct feature of this expression is the alternating sign pattern. We want to preserve this feature as we increase matrix size.

Definition 1. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Define the *determinant* of A by

$$\begin{aligned} \det A &= (-1)^{1+1} a_{11} \det A_{11} + (-1)^{1+2} a_{12} \det A_{12} + \dots \\ &\quad \dots + (-1)^{1+j} a_{1j} \det A_{1j} + \dots \\ &\quad \dots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

Naturally, we would like to condense this formula. To accomplish this, let

$$C_{1j} = (-1)^{1+j} \det A_{1j}$$

We will refer to C_{1j} as the $(1, j)$ -cofactor of A . When we use the cofactor notation, the expression in Definition ?? turns into the following:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n} = \sum_{j=1}^n a_{1j}C_{1j}$$

This process of computing the determinant is called the *cofactor expansion along the first row*.

Cofactor Expansion Along the First Column

As we have observed in several examples in [Finding the Determinant](#), cofactor expansion along the first column produces the same result as cofactor expansion along the top row. We will now formalize the process of cofactor expansion along the first column for an $n \times n$ matrix and prove that this process produces the same result as our original definition of the determinant. Let A be an $n \times n$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Define A_{i1} to be an $(n-1) \times (n-1)$ matrix obtained from A by deleting the first column and the i^{th} row of A . We say that A_{i1} is the $(i, 1)$ -minor of A .

$$\begin{bmatrix} a_{11} & \boxed{a_{12} \dots a_{1n}} \\ a_{21} & \boxed{a_{22} \dots a_{2n}} \\ \vdots & \vdots \\ a_{i1} & \boxed{a_{i2} \dots a_{in}} \\ \vdots & \vdots \\ a_{n1} & \boxed{a_{n2} \dots a_{nn}} \end{bmatrix}$$

Define $C_{i1} = (-1)^{i+1} \det A_{i1}$ to be the $(i, 1)$ -cofactor of A .

Definition 2. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Define the *determinant* of A by

$$\begin{aligned} \det A &= (-1)^{1+1} a_{11} \det A_{11} + (-1)^{2+1} a_{21} \det A_{21} + \dots \\ &\quad \dots + (-1)^{i+1} a_{i1} \det A_{i1} + \dots \\ &\quad \dots + (-1)^{n+1} a_{n1} \det A_{n1} \\ &= \sum_{i=1}^n (-1)^{i+1} a_{i1} \det A_{i1} \end{aligned}$$

or

$$\det A = a_{11}C_{11} + a_{21}C_{21} + \dots + a_{n1}C_{n1} = \sum_{i=1}^n a_{i1}C_{i1}$$

Proof of Definition Equivalence

We will now show that cofactor expansion along the first row produces the same result as cofactor expansion along the first column.

Theorem 1. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j} = \sum_{i=1}^n (-1)^{i+1} a_{i1} \det A_{i1}$$

Proof We will proceed by induction on n . Clearly, the result holds for $n = 1$. Just for practice you should also verify the equality for $n = 2, 3$. (See Practice Problem ??.) We will assume that the result holds for $(n-1) \times (n-1)$ matrices and show that it must hold for $n \times n$ matrices.

You will find the following matrix a useful reference as we proceed.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

For convenience, we will refer to the Right-Hand Side (RHS) and the Left-Hand Side (LHS) of the equality we are trying to prove.

$$\text{LHS} = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j} \stackrel{?}{=} \sum_{i=1}^n (-1)^{i+1} a_{i1} \det A_{i1} = \text{RHS}$$

Note that the first term $a_{11}(-1)^{1+1} \det A_{11}$ is the same for LHS and RHS, so we will only need to consider $i, j \geq 2$.

We will start by analyzing RHS. Consider an arbitrary entry a_{i1} of the first column. This entry will only appear in the term $a_{i1}(-1)^{i+1} \det A_{i1}$. We will find $\det A_{i1}$ by cofactor expansion along the first row. As we proceed, we have to pay special attention to the subscripts. Because the first column of A was removed, the j^{th} column of A contains the $(j-1)$ column of A_{i1} .

$$\begin{aligned} a_{i1}(-1)^{i+1} \det A_{i1} &= \\ &= a_{i1}(-1)^{i+1} \left(a_{12}(-1)^{1+1} \det (A_{i1})_{11} + \dots + a_{1j}(-1)^{1+(j-1)} \det (A_{i1})_{1(j-1)} + \dots \right) \end{aligned}$$

Note that the entry a_{1j} will only appear in the term

$$a_{1j}(-1)^{1+(j-1)} \det (A_{i1})_{1(j-1)}$$

So, after we distribute $a_{i1}(-1)^{i+1}$, RHS will contain only one term of the form

$$a_{i1} a_{1j} (-1)^{p+q} \det (A_{st})_{pq}$$

We will perform a similar analysis on LHS. Consider an arbitrary entry a_{1j} of the first row. This entry will only appear in the term $a_{1j}(-1)^{1+j}(\det A_{1j})$. Invoking the induction hypothesis, we will find $\det A_{1j}$ by cofactor expansion along the first column.

$$\begin{aligned} & a_{1j}(-1)^{1+j} \det A_{1j} = \\ & = a_{1j}(-1)^{1+j} \left(a_{21}(-1)^{1+1} \det (A_{1j})_{11} + \dots + a_{i1}(-1)^{(i-1)+1} \det (A_{1j})_{(i-1)1} + \dots \right) \end{aligned}$$

The entry a_{i1} will only appear in the term

$$a_{i1}(-1)^{(i-1)+1} \det (A_{1j})_{(i-1)1}$$

So, after we distribute $a_{1j}(-1)^{1+j}$, LHS will contain only one term of the form

$$a_{i1} a_{1j}(-1)^{p+q} \det (A_{st})_{pq}$$

But RHS also has only one term of this form. We now need to show that these two terms are equal. The two terms are

$$a_{i1}(-1)^{i+1} a_{1j}(-1)^{1+(j-1)} \det (A_{i1})_{1(j-1)} = a_{i1} a_{1j}(-1)^{i+j+1} \det (A_{i1})_{1(j-1)}$$

and

$$a_{1j}(-1)^{1+j} a_{i1}(-1)^{(i-1)+1} \det (A_{1j})_{(i-1)1} = a_{i1} a_{1j}(-1)^{i+j+1} \det (A_{1j})_{(i-1)1}$$

Observe that $(A_{i1})_{1(j-1)}$ and $(A_{1j})_{(i-1)1}$ are the same matrix because both were obtained from matrix A by deleting the first and the i^{th} rows of A , and the first and the j^{th} columns of A . Therefore $\det (A_{i1})_{1(j-1)} = \det (A_{1j})_{(i-1)1}$.

We conclude that the terms of LHS and RHS match. This establishes the desired equality. ■

Now we know that cofactor expansion along the first row and cofactor expansion along the first column produce the same result, so either expansion can be used to find the determinant.

Proof of Results Concerning Elementary Row Operations

In [Elementary Row Operations](#) we observed, without proof, the following properties of the determinant. (See Theorem ??.)

Summary 1. Let $A = [a_{ij}]$ be an $n \times n$ matrix.

1. If B is obtained from A by interchanging two different rows, then

$$\det B = -\det A$$

2. If B is obtained from A by multiplying one of the rows of A by a non-zero constant k . Then

$$\det B = k \det A$$

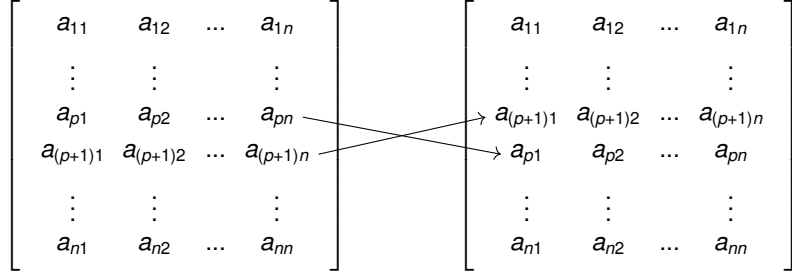
3. If B is obtained from A by adding a multiple of one row of A to another row, then

$$\det B = \det A$$

We now prove these properties.

Proof of Theorem ???? We will start by showing that the result holds if two consecutive rows are interchanged. Suppose B is obtained from A by swapping rows p and $p + 1$ of A .

We proceed by induction on n . The result is not applicable for $n = 1$. In Practice Problem ??, you will be asked to verify that the result holds for 2×2 matrices. Suppose that the result holds for $(n - 1) \times (n - 1)$ matrices. We need to show that it holds for $n \times n$ matrices. You may find the following diagram useful throughout the proof.



Observe that for $i \neq p, p + 1$ we have:

$$b_{ij} = a_{ij}$$

Because B_{ij} is obtained from A_{ij} by switching two rows of A_{ij} , our induction hypothesis give us:

$$\det B_{ij} = -\det A_{ij}$$

For $i = p$ and $i = p + 1$ we have:

$$b_{p1} = a_{(p+1)1} \quad \text{and} \quad b_{(p+1)1} = a_{p1}$$

$$B_{p1} = A_{(p+1)1} \quad \text{and} \quad B_{(p+1)1} = A_{p1}$$

We compute the determinant of B by cofactor expansion along the first column.

$$\begin{aligned} \det B &= b_{11}(-1)^{1+1} \det B_{11} + \dots + b_{i1}(-1)^{i+1} \det B_{i1} + \dots \\ &\quad + b_{p1}(-1)^{p+1} \det B_{p1} + b_{(p+1)1}(-1)^{(p+1)+1} \det B_{(p+1)1} + \dots \\ &\quad + b_{n1}(-1)^{n+1} \det B_{n1} \\ &= a_{11}(-1)^{1+1}(-1) \det A_{11} + \dots + a_{i1}(-1)^{i+1}(-1) \det A_{i1} + \dots \\ &\quad + a_{(p+1)1}(-1)^{p+1} \det A_{(p+1)1} + a_{p1}(-1)^{(p+1)+1} \det A_{p1} + \dots \\ &\quad + a_{n1}(-1)^{n+1}(-1) \det A_{n1} \\ &= a_{11}(-1)^{1+1}(-1) \det A_{11} + \dots + a_{i1}(-1)^{i+1}(-1) \det A_{i1} + \dots \\ &\quad + a_{(p+1)1}(-1)^{(p+1)+1}(-1) \det A_{(p+1)1} + a_{p1}(-1)^{p+1}(-1) \det A_{p1} + \dots \\ &\quad + a_{n1}(-1)^{n+1}(-1) \det A_{n1} \\ &= (-1) \left(a_{11}(-1)^{1+1} \det A_{11} + \dots + a_{i1}(-1)^{i+1} \det A_{i1} + \dots \right) \end{aligned}$$

$$\begin{aligned}
& + a_{(p+1)1}(-1)^{(p+1)+1} \det A_{(p+1)1} + a_{p1}(-1)^{p+1} \det A_{p1} + \dots \\
& + a_{n1}(-1)^{n+1} \det A_{n1} \Big) \\
& = -\det A
\end{aligned}$$

If two non-adjacent rows are switched, then the switch can be carried out by performing an odd number of adjacent row interchanges (See Practice Problem ??), so the result still holds. ■

Proof of Theorem ???? ? We proceed by induction on n . Clearly the statement is true for $n = 1$. Just for fun, you might want to verify directly that it holds for 2×2 matrices. Now suppose the statement is true for all $(n-1) \times (n-1)$ matrices. We will show that it holds for $n \times n$ matrices.

Suppose B is obtained from A by multiplying the p 's row of A by k .

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & \dots & a_{pn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{p1} & ka_{p2} & \dots & ka_{pn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

We compute the determinant of B by cofactor expansion along the first column.

$$\begin{aligned}
\det B &= a_{11}(-1)^{1+1} \det B_{11} + \dots + ka_{p1}(-1)^{p+1} \det B_{p1} + \dots \\
&+ a_{n1}(-1)^{n+1} \det B_{n1} \\
&= ka_{11}(-1)^{1+1} \det A_{11} + \dots + ka_{p1}(-1)^{p+1} \det A_{p1} + \dots \\
&+ ka_{n1}(-1)^{n+1} \det A_{n1} \\
&= k \left(a_{11}(-1)^{1+1} \det A_{11} + \dots + a_{p1}(-1)^{p+1} \det A_{p1} + \dots \right. \\
&\quad \left. + a_{n1}(-1)^{n+1} \det A_{n1} \right) \\
&= k \det A
\end{aligned}$$

■

Before we tackle the proof of Part ?? of Theorem ?? we will need to prove the following lemma.

Lemma 1. Let A , B and C be $n \times n$ matrices. Suppose A , B and C are identical, except for the p^{th} row. If the p^{th} row of C is the sum of the p^{th} rows of A and B , then

$$\det C = \det A + \det B$$

Proof We will proceed by induction on n . We leave it to the reader to verify cases $n = 1, 2$. We will assume that the statement holds for all $(n - 1) \times (n - 1)$ matrices and show that it holds for $n \times n$ matrices.

You may find the following representations of A , B and C helpful. Identical entries in A , B and C are labeled d_{ij} .

$$A = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & \dots & a_{pn} \\ \vdots & \vdots & & \vdots \\ d_{n1} & d_{n2} & \dots & d_{nn} \end{bmatrix} \quad B = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pn} \\ \vdots & \vdots & & \vdots \\ d_{n1} & d_{n2} & \dots & d_{nn} \end{bmatrix}$$

$$C = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ \vdots & \vdots & & \vdots \\ a_{p1} + b_{p1} & a_{p2} + b_{p2} & \dots & a_{pn} + b_{pn} \\ \vdots & \vdots & & \vdots \\ d_{n1} & d_{n2} & \dots & d_{nn} \end{bmatrix}$$

Observe that

$$A_{p1} = B_{p1} = C_{p1}$$

For $i \neq p$, the induction hypothesis gives us

$$\det C_{i1} = \det A_{i1} + \det B_{i1}$$

We now compute the determinant of C by cofactor expansion along the first column.

$$\begin{aligned} \det C &= d_{11}(-1)^{1+1} \det C_{11} + \dots + (a_{p1} + b_{p1})(-1)^{p+1} \det C_{p1} + \dots \\ &\quad + d_{n1}(-1)^{n+1} \det C_{n1} \\ &= d_{11}(-1)^{1+1} (\det A_{11} + \det B_{11}) + \dots \\ &\quad + a_{p1}(-1)^{p+1} \det C_{p1} + b_{p1}(-1)^{p+1} \det C_{p1} + \dots \\ &\quad + d_{n1}(-1)^{n+1} (\det A_{n1} + \det B_{n1}) \\ &= d_{11}(-1)^{1+1} \det A_{11} + d_{11}(-1)^{1+1} \det B_{11} + \dots \\ &\quad + a_{p1}(-1)^{p+1} \det A_{p1} + b_{p1}(-1)^{p+1} \det B_{p1} + \dots \\ &\quad + d_{n1}(-1)^{n+1} \det A_{n1} + d_{n1}(-1)^{n+1} \det B_{n1} \\ &= \det A + \det B \end{aligned}$$

■

We are now ready to finish the proof of Theorem ??

Proof of Theorem ????? Suppose B is obtained from A by adding k times row p to row q . ($p \neq q$ and $k \neq 0$) You may find the following representations of A and B useful.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & \dots & a_{pn} \\ \vdots & \vdots & & \vdots \\ a_{q1} & a_{q2} & \dots & a_{qn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & \dots & a_{pn} \\ \vdots & \vdots & & \vdots \\ a_{q1} + ka_{p1} & a_{q2} + ka_{p2} & \dots & a_{qn} + ka_{pn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

We will form another matrix A' by replacing the q^{th} row of A with k times the p^{th} row.

$$A' = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & \dots & a_{pn} \\ \vdots & \vdots & & \vdots \\ ka_{p1} & ka_{p2} & \dots & ka_{pn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Observe that matrices A , A' and B are identical except for the q^{th} row, and the q^{th} row of matrix B is the sum of the q^{th} rows of A and A' . Thus, by Lemma ??, we have:

$$\det B = \det A + \det A'$$

Since one row of A' is a scalar multiple of another row, we know $\det A' = 0$ (see Practice Problem ??). Therefore $\det B = \det A$. ■

The Laplace Expansion Theorem

As we have seen in examples, the value of the determinant can be computed by expanding along any row or column. This result is known as the Laplace Expansion Theorem. We begin by generalizing some earlier definitions.

Given an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(j-1)} & a_{2j} & a_{2(j+1)} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \dots & a_{(i-1)(j-1)} & a_{(i-1)j} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ a_{i1} & a_{i2} & \dots & a_{i(j-1)} & a_{ij} & a_{i(j+1)} & \dots & a_{in} \\ a_{(i+1)1} & a_{(i+1)2} & \dots & a_{(i+1)(j-1)} & a_{(i+1)j} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(j-1)} & a_{nj} & a_{n(j+1)} & \dots & a_{nn} \end{bmatrix}$$

define the *minor* A_{ij} to be an $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column of A .

$$\left[\begin{array}{cccc|c|cccc} a_{11} & a_{12} & \dots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(j-1)} & a_{2j} & a_{2(j+1)} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \dots & a_{(i-1)(j-1)} & a_{(i-1)j} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ a_{i1} & a_{i2} & \dots & a_{ij(j-1)} & a_{ij} & a_{i(j+1)} & \dots & a_{in} \\ a_{(i+1)1} & a_{(i+1)2} & \dots & a_{(i+1)(j-1)} & a_{(i+1)j} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(j-1)} & a_{nj} & a_{n(j+1)} & \dots & a_{nn} \end{array} \right]$$

Define the (i, j) -cofactor of A by

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

Note that the sign of $(-1)^{i+j}$ follows a checkerboard pattern.

$$\begin{bmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Theorem 2 (Laplace Expansion Theorem). Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then each of the following computations produces $\det A$.

1. *Cofactor Expansion along the i^{th} row*

$$\begin{aligned} \det A &= a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in} \\ &= a_{i1} (-1)^{i+1} \det A_{i1} + a_{i2} (-1)^{i+2} \det A_{i2} + \dots + a_{in} (-1)^{i+n} \det A_{in} \\ &= \sum_{j=1}^n a_{ij} (-1)^{i+j} \det A_{ij} \end{aligned}$$

2. *Cofactor Expansion along the j^{th} column*

$$\begin{aligned} \det A &= a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj} \\ &= a_{1j} (-1)^{1+j} \det A_{1j} + a_{2j} (-1)^{2+j} \det A_{2j} + \dots + a_{nj} (-1)^{n+j} \det A_{nj} \\ &= \sum_{i=1}^n a_{ij} (-1)^{i+j} \det A_{ij} \end{aligned}$$

Proof We will start by showing that cofactor expansion along column j produces the same result as cofactor expansion along the first column. Observe that column j can be shifted into the first column position by $j-1$

consecutive row switches. Let $A' = [a'_{ij}]$ be the matrix obtained from A by performing the necessary column switches. Then

$$\begin{aligned}
 \det A &= (-1)^{j-1} \det A' \\
 &= (-1)^{j-1} \sum_{i=1}^n a'_{i1} (-1)^{i+1} \det A'_{i1} \\
 &= (-1)^{j-1} \sum_{i=1}^n a_{ij} (-1)^{i+1} \det A_{ij} \\
 &= \sum_{i=1}^n a_{ij} (-1)^{j-1} (-1)^{i+1} \det A_{ij} \\
 &= \sum_{i=1}^n a_{ij} (-1)^{i+j} \det A_{ij}
 \end{aligned}$$

To show that the determinant of A can also be computed by cofactor expansion along any row follows from the fact that $\det A = \det A^T$. (Theorem ??) ■

Practice Problems

Problem 1 Complete the proof of Theorem ???? by showing that the result holds for a 2×2 matrix.

Problem 2 Let p and q be two rows of a matrix, with $p < q$. Show that the switch of p and q requires $2(q - p) - 1$ adjacent row interchanges.