

1 Determinants and Inverses of Nonsingular Matrices

Combining results of Theorem ?? and Theorem ?? shows that the following statements about matrix A are equivalent:

- A^{-1} exists
- Any equation $Ax = \mathbf{b}$ has a unique solution
- $\det A \neq 0$

In this section we will take a closer look at the relationship between the determinant of a nonsingular matrix A , solution to the system $A\mathbf{x} = \mathbf{b}$, and the inverse of A .

Cramer's Rule

We begin by establishing a formula that allows us to express the unique solution to the system $\mathbf{Ax} = \mathbf{b}$ in terms of the determinant of A , for a nonsingular matrix A . This formula is called *Cramer's rule*.

Consider the system

$$\begin{array}{rcl} ax & + & by = e \\ cx & + & dy = f \end{array}$$

The system can be written as a matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

Using one of our standard methods for solving systems we find that

$$x = \frac{ed - bf}{ad - bc} \quad \text{and} \quad y = \frac{af - ec}{ad - bc}$$

Observe that the denominators in the expressions for x and y are the same and equal to $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

A close examination shows that the numerators of expressions for x and y can also be interpreted as determinants of matrices. The numerator of the expression for x is the determinant of the matrix that is formed by replacing the first column of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\begin{bmatrix} e \\ f \end{bmatrix}$. The numerator of the expression for y is the determinant of the matrix that is formed by

replacing the second column of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\begin{bmatrix} e \\ f \end{bmatrix}$. Thus, x and y can be written as

$$x = \frac{ed - bf}{ad - bc} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y = \frac{af - ec}{ad - bc} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Note that a unique solution to the system exists if and only if the determinant of the coefficient matrix is not zero.

It turns out that a solution to any square system $A\mathbf{x} = \mathbf{b}$ can be expressed using ratios of determinants, provided that A is nonsingular. The general formula for the i^{th} component of the solution vector is

$$x_i = \frac{\det(\text{matrix } A \text{ with column } i \text{ replaced by } \mathbf{b})}{\det A}$$

To formalize this expression, we need to introduce some notation. Given a matrix

$$A = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \\ | & | & & | \end{bmatrix}$$

and a vector $\mathbf{b}$ we use

$$A_i(\mathbf{b})$$

to denote the matrix obtained from A by replacing the i^{th} column of A with $\mathbf{b}$. In other words,

$$A_i(\mathbf{b}) = \begin{bmatrix} | & | & & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_{i-1} & \mathbf{b} & \mathbf{a}_{i+1} & \dots & \mathbf{a}_n \\ | & | & & | & | & | & | \end{bmatrix} \quad (1)$$

Using our new notation, we can write the i^{th} component of the solution vector as

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}$$

We will work through a couple of examples before proving this result as a theorem.

Example 1. Solve $A\mathbf{x} = \mathbf{b}$ using Cramer's rule if

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Explanation. We start by computing the determinant of A .

$$\det A = \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} = (3)(5) - (-1)(2) = 17$$

Next, we compute $\det A_1(\mathbf{b})$ and $\det A_2(\mathbf{b})$.

$$\det A_1(\mathbf{b}) = \begin{vmatrix} -2 & -1 \\ 4 & 5 \end{vmatrix} = (-2)(5) - (-1)(4) = -6$$

$$\det A_2(\mathbf{b}) = \begin{vmatrix} 3 & -2 \\ 2 & 4 \end{vmatrix} = (3)(4) - (-2)(2) = 16$$

We now compute the components of the solution vector.

$$x_1 = \frac{-6}{17} \quad \text{and} \quad x_2 = \frac{16}{17}$$

Finally, it is a good idea to verify that what we found is a solution to the system.

$$\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -6/17 \\ 16/17 \end{bmatrix} = \begin{bmatrix} -18/17 - 16/17 \\ -12/17 + 80/17 \end{bmatrix} = \begin{bmatrix} -34/17 \\ 68/17 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Example 2. Solve $A\mathbf{x} = \mathbf{b}$ using Cramer's rule if

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

Explanation. Find the determinant of A .

$$\det A = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = \dots \dots$$

Next, we compute $\det A_i(\mathbf{b})$ for $i = 1, 2, 3$.

$$\det A_1(\mathbf{b}) = \begin{vmatrix} -2 & 2 & -1 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \dots \dots$$

$$\det A_2(\mathbf{b}) = \begin{vmatrix} 1 & -2 & -1 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \dots \dots$$

$$\det A_3(\mathbf{b}) = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & 3 \\ 0 & 3 & 1 \end{vmatrix} = \dots \dots$$

This gives us the solution vector

$$\mathbf{x} = \begin{bmatrix} \dots \dots \\ \dots \dots \\ \dots \dots \end{bmatrix}$$

You should verify that what you found really is a solution.

We are now ready to state and prove Cramer's rule as a theorem.

Theorem 1. Let A be a nonsingular $n \times n$ matrix, and let $\mathbf{b}$ be an $n \times 1$ vector. Then the components of the solution vector $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ are given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}$$

where $A_i(\mathbf{b})$ is defined as in (??).

Proof For this proof we will need to think of matrices in terms of their columns. Thus,

$$A = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \\ | & | & & | \end{bmatrix}$$

We will also need the identity matrix I . The columns of I are standard unit vectors.

$$I = \begin{bmatrix} | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ | & | & & | \end{bmatrix}$$

Recall that

$$A_i(\mathbf{b}) = \begin{bmatrix} | & | & & | & | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_{i-1} & \mathbf{b} & \mathbf{a}_{i+1} & \dots & \mathbf{a}_n \\ | & | & & | & | & | & & | \end{bmatrix}$$

Similarly,

$$I_i(\mathbf{x}) = \begin{bmatrix} | & | & & | & | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_{i-1} & \mathbf{x} & \mathbf{e}_{i+1} & \dots & \mathbf{e}_n \\ | & | & & | & | & | & & | \end{bmatrix}$$

Observe that x_i is the only non-zero entry in the i^{th} row of $I_i(\mathbf{x})$. Cofactor expansion along the i^{th} row gives us

$$\det I_i(\mathbf{x}) = x_i \tag{2}$$

Now, consider the product $A(I_i(\mathbf{x}))$

$$\begin{aligned} A(I_i(\mathbf{x})) &= \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} | & | & & | & | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_{i-1} & \mathbf{x} & \mathbf{e}_{i+1} & \dots & \mathbf{e}_n \\ | & | & & | & | & | & & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & & | & | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_{i-1} & A\mathbf{x} & \mathbf{a}_{i+1} & \dots & \mathbf{a}_n \\ | & | & & | & | & | & & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & & | & | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_{i-1} & \mathbf{b} & \mathbf{a}_{i+1} & \dots & \mathbf{a}_n \\ | & | & & | & | & | & & | \end{bmatrix} = A_i(\mathbf{b}) \end{aligned}$$

This gives us

$$A I_i(\mathbf{x}) = A_i(\mathbf{b})$$

$$\det A I_i(\mathbf{x}) = \det A_i(\mathbf{b})$$

$$\det A \det I_i(\mathbf{x}) = \det A_i(\mathbf{b})$$

By our earlier observation in (??) we have

$$\det A x_i = \det A_i(\mathbf{b})$$

A is nonsingular, so $\det A \neq 0$. Thus

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}$$

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Finding the determinant is computationally expensive. Because Cramer's rule requires finding many determinants, it is not a computationally efficient way of solving a system of equations. However, Cramer's rule is often used for small systems in applications that arise in economics, natural, and social sciences, particularly when solving for only a subset of the variables.

Adjugate Formula for the Inverse of a Matrix

In Practice Problem ?? we used the row reduction algorithm to show that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is nonsingular then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (3)$$

This formula is a special case of a general formula for the inverse of a nonsingular square matrix. Just like the formula for a 2×2 matrix, the general formula includes the coefficient $\frac{1}{\det A}$ and a matrix related to the original matrix. We will now derive the general formula using Cramer's rule.

Let A be an $n \times n$ nonsingular matrix. When looking for the inverse of A , we look for a matrix X such that $AX = I$. We will think of matrices in terms of their columns

$$I = \begin{bmatrix} | & | & \dots & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ | & | & \dots & | \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} | & | & \dots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \\ | & | & \dots & | \end{bmatrix}$$

If $AX = I$ then we must have

$$A\mathbf{x}_1 = \mathbf{e}_1$$

$$A\mathbf{x}_2 = \mathbf{e}_2$$

$$\vdots$$

$$A\mathbf{x}_n = \mathbf{e}_n$$

This gives us n systems of equations. Solution vectors to these systems are the columns of X . Thus, the j^{th} column of X is

$$\mathbf{x}_j = \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{bmatrix} \quad \text{such that} \quad A\mathbf{x}_j = \mathbf{e}_j$$

By Cramer's rule

$$x_{ij} = \frac{\det A_i(\mathbf{e}_j)}{\det A}$$

But

$$A_i(\mathbf{e}) = \begin{bmatrix} | & | & & | & | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_{i-1} & \mathbf{e}_j & \mathbf{a}_{i+1} & \dots & \mathbf{a}_n \\ | & | & & | & | & | & & | \end{bmatrix}$$

To find $\det A_i(\mathbf{e}_j)$, we can expand along the j^{th} column of $A_i(\mathbf{e}_j)$. But the j^{th} column of $A_i(\mathbf{e}_j)$ is the vector $\mathbf{e}_j$ which has 1 in the j^{th} spot and zeros everywhere else. Thus

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$$

We now have

$$\mathbf{x}_j = \begin{bmatrix} C_{j1}/\det A \\ C_{j2}/\det A \\ \vdots \\ C_{jn}/\det A \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} C_{j1} \\ C_{j2} \\ \vdots \\ C_{jn} \end{bmatrix}$$

Thus,

$$A^{-1} = X = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

The matrix of cofactors of A is called the *adjugate* of A . We write

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

Warning 1. Note the order of subscripts of C in the adjugate matrix. The (i, j) -entry of the adjugate matrix is C_{ji} .

We summarize our result as a theorem

Theorem 2. Let A be a nonsingular square matrix, then

$$A^{-1} = \frac{1}{\det A} \text{adj}(A)$$

Example 3. Use Theorem ?? to find A^{-1} if

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Explanation. We begin by finding $\det A$

$$\det A = \dots\dots$$

The first column of $\text{adj}(A)$ has entries C_{11} , C_{12} and C_{13} .

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = \dots\dots$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = \dots\dots$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = \dots\dots$$

The second column of $\text{adj}(A)$ has entries C_{21} , C_{22} and C_{23} .

$$C_{21} = (-1)^{2+1} \begin{vmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{vmatrix} = \dots\dots$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{vmatrix} = \dots\dots$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \end{vmatrix} = \dots\dots$$

Now we compute the third column of $\text{adj}(A)$.

$$C_{31} = \dots\dots$$

$$C_{32} = \dots\dots$$

$$C_{33} = \dots\dots$$

This gives us

$$A^{-1} = \begin{bmatrix} 2 & -5/2 & 3/2 \\ -1 & 3/2 & -1/2 \\ -1 & 2 & -1 \end{bmatrix}$$

Compare this result to the answer in Problem ??.

Practice Problems

Problems ??-??

Use Cramer's rule to solve each of the following systems.

Problem 1

$$\begin{array}{rcrcrcrcl} -3x & + & 2y & = & 1 \\ 2x & - & y & = & 4 \end{array}$$

Answer:

$$x = \dots\dots \quad y = \dots\dots$$

Problem 2

$$\begin{bmatrix} 2 & -5 & 1 \\ 6 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

Answer:

$$\mathbf{x} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 3 Consider the equation

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & -1 \\ 3 & -2 & 3 & -2 \\ 2 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 0 \\ 11 \end{bmatrix}$$

1. Solve for x_2 using Cramer's Rule.

Answer:

$$x_2 = \dots$$

2. If you had to solve for all four variables, which method would you use? Why?

Problems ??-??

Use Theorem ?? to find the inverse of each of the following matrices.

Problem 4

$$A = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$$

Answer:

$$A^{-1} = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Problem 5

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 4 & -2 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

Answer:

$$A^{-1} = \frac{1}{\dots} \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Problem 6 Show that the formula in (??) is a special case of the formula in Theorem ?? by showing that

$$\text{adj} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$