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# 1 Similar Matrices and Their Properties

Let  $A$  and  $B$  be  $n \times n$  matrices. Then the products  $AB$  and  $BA$  are both  $n \times n$  matrices. In most cases the products  $AB$  and  $BA$  are not equal.

However, for some pairs of  $n \times n$  matrices  $A$  and  $B$ , we are able to find an invertible matrix  $P$  such that  $PB = AP$ . This leads to the following definition.

**Definition 1.** If  $A$  and  $B$  are  $n \times n$  matrices, we say that  $A$  and  $B$  are *similar*, if  $B = P^{-1}AP$  for some invertible matrix  $P$ . In this case we write  $A \sim B$ .

The following theorem shows that *similarity* ( $\sim$ ) satisfies *reflexive*, *symmetric*, and *transitive* properties.

**Theorem 1.** Similarity is an *equivalence relation*, i.e. for  $n \times n$  matrices  $A$ ,  $B$ , and  $C$ ,

1.  $A \sim A$  (reflexive)
2. If  $A \sim B$ , then  $B \sim A$  (symmetric)
3. If  $A \sim B$  and  $B \sim C$ , then  $A \sim C$  (transitive)

**Proof** ?? It is clear that  $A \sim A$  (let  $P = I$ ).

?? If  $A \sim B$ , then for some invertible matrix  $P$ ,

$$A = P^{-1}BP$$

and so

$$PAP^{-1} = B$$

But then

$$(P^{-1})^{-1}AP^{-1} = B$$

which shows that  $B \sim A$ .

?? Now suppose  $A \sim B$  and  $B \sim C$ . Then there exist invertible matrices  $P, Q$  such that

$$A = P^{-1}BP, \quad B = Q^{-1}CQ$$

Then,

$$A = P^{-1}(Q^{-1}CQ)P = (QP)^{-1}C(QP)$$

showing that  $A$  is similar to  $C$ . ■

Any relation satisfying the reflexive, symmetric and transitive properties is called an *equivalence relation*. Theorem ?? proves that similarity between matrices is an *equivalence relation*. Practice Problem ?? gives a good example of a relation that is NOT an equivalence relation.

As we will see later, similar matrices share many properties. Before proceeding to explore these properties, we pause to introduce a simple matrix function that we will continue to use throughout the course.

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**Definition 2.** The *trace* of an  $n \times n$  matrix  $A$ , abbreviated  $\text{tr}A$ , is defined to be the sum of the main diagonal elements of  $A$ . In other words, if  $A = [a_{ij}]$ , then

$$\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

We may also write  $\text{tr}(A) = \sum_{i=1}^n a_{ii}$ .

It is easy to see that  $\text{tr}(A + B) = \text{tr}A + \text{tr}B$  and that  $\text{tr}(cA) = c \text{tr}(A)$  holds for all  $n \times n$  matrices  $A$  and  $B$  and all scalars  $c$ . The following fact is more surprising.

**Theorem 2.** Let  $A$  and  $B$  be  $n \times n$  matrices. Then  $\text{tr}(AB) = \text{tr}(BA)$ .

**Proof** Write  $A = [a_{ij}]$  and  $B = [b_{ij}]$ . For each  $i$ , the  $(i, i)$ -entry  $d_i$  of the matrix  $AB$  is given as follows:  $d_i = a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni} = \sum_j a_{ij}b_{ji}$ . Hence

$$\text{tr}(AB) = d_1 + d_2 + \cdots + d_n = \sum_i d_i = \sum_i \left( \sum_j a_{ij}b_{ji} \right)$$

Similarly we have  $\text{tr}(BA) = \sum_i \left( \sum_j b_{ij}a_{ji} \right)$ . Since these two double sums are the same, we have proven the theorem. ■

The following theorem lists a number of properties shared by similar matrices.

**Theorem 3.** If  $A$  and  $B$  are  $n \times n$  matrices and  $A \sim B$ , then

1.  $\det(A) = \det(B)$ ,
2.  $\text{rank}(A) = \text{rank}(B)$ ,
3.  $\text{tr}(A) = \text{tr}(B)$ ,
4.  $A$  and  $B$  have the same characteristic equations, and
5.  $A$  and  $B$  have the same eigenvalues.

**Proof** Let  $B = P^{-1}AP$  for some invertible matrix  $P$ .

For ??,  $\det B = \det(P^{-1}) \det A \det P = \det A$  because  $\det(P^{-1}) = 1 / \det P$  (Theorem ??).

Similarly, for ??  $\text{rank} B = \text{rank}(P^{-1}AP) = \text{rank} A$ , because multiplication by an invertible matrix cannot change the rank. To see this, note that any invertible matrix is a product of elementary matrices. Multiplying by elementary matrices is equivalent to performing elementary row (column) operations on  $A$ , which does not change the row (column) space, nor the rank. It follows that similar matrices have the same rank.

For ??, we make use of Theorem ??:

$$\text{tr}(P^{-1}AP) = \text{tr}[P^{-1}(AP)] = \text{tr}[(AP)P^{-1}] = \text{tr}A.$$

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As for ??,

$$\begin{aligned}\det(B - \lambda I) &= \det\{P^{-1}AP - \lambda(P^{-1}P)\} \\ &= \det\{P^{-1}(A - \lambda I)P\} \\ &= \det(A - \lambda I),\end{aligned}$$

so  $A$  and  $B$  have the same characteristic equation. Finally, ?? implies ?? because the eigenvalues of a matrix are the roots of its characteristic polynomial. ■

**Remark 1.** Sharing the five properties in Theorem ?? does not guarantee that two matrices are similar. The matrices  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  and  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  have the same determinant, rank, trace, characteristic polynomial, and eigenvalues, but they are not similar because  $P^{-1}IP = I$  for any invertible matrix  $P$ .

Even though the properties in Theorem ?? cannot be used to show two matrices are similar, these properties come in handy for showing that two matrices are NOT similar.

**Example 1.** Are the matrices  $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}$  similar?

**Explanation.** A quick check shows us  $\det A = \det B$ , and both matrices are seen to be invertible, so they have the same rank. However,  $\text{tr}A = 1$  and  $\text{tr}B = 2$ , so the matrices are not similar.

The next theorem shows that similarity is preserved under inverses, transposes, and powers:

**Theorem 4.** If  $A$  and  $B$  are  $n \times n$  matrices and  $A \sim B$ , then

1.  $A^{-1} \sim B^{-1}$ ,
2.  $A^T \sim B^T$ , and
3.  $A^k \sim B^k$  for all integers  $k \geq 1$ .

**Proof** See Practice Problem ??. ■

## Practice Problems

**Problem 1** At the beginning of this section we mentioned that similarity of  $n \times n$  matrices is an equivalence relation.

An equivalence relation is a binary relation  $R$  on elements of a set  $S$  that has the following properties:

- The reflexive property:  $xRx$  for every  $x \in S$
- The symmetric property: If  $xRy$ , then  $yRx$  for every  $x, y \in S$
- The transitive property: If  $xRy$  and  $yRz$ , then  $xRz$  for every  $x, y, z \in S$

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Let  $S$  be the set of all positive integers. We can show that the relation “less than” (symbolized by  $<$ ) is NOT an equivalence relation on this set. To see this, note that “less than” is not reflexive, because  $s < s$  is not true for any positive integer  $s$ .

1. Is the relation “less than” symmetric? (Yes/ No)

2. Is the relation “less than” transitive? (Yes/ No)

**Problem 2** Another relation between matrices we have studied in this course is that two matrices can be “row equivalent”. Is the relation “row equivalent”

1. reflexive? (Yes/ No)

2. symmetric? (Yes/ No)

3. transitive? (Yes/ No)

**Problem 3** Use Theorem ?? to show that  $A$  and  $B$  are not similar.  $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

**Problem 4** Use Theorem ?? to show that  $A$  and  $B$  are not similar.  $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

**Problem 5** Use Theorem ?? to show that  $A$  and  $B$  are not similar.  $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -1 & 2 \\ 0 & 3 & -5 \end{bmatrix}$ ,  $B = \begin{bmatrix} -2 & 1 & 3 \\ 6 & -3 & -9 \\ 0 & 0 & 0 \end{bmatrix}$

**Problem 6** Show that  $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 4 & 3 & 0 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 & -1 & 3 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 4 & 1 \\ 5 & -1 & -1 & -4 \end{bmatrix}$  are not similar.

**Problem 7** Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

**Question 7.1** The eigenvalues of  $A$  are known to be  $\lambda_1 = -3$ ,  $\lambda_2 = 3$  and  $\lambda_3 = 1$ . Find the corresponding eigenspaces.

$$S_{-3} = \text{span} \left( \begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

$$S_3 = \text{span} \left( \begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

$$S_1 = \text{span} \left( \begin{bmatrix} \dots \\ \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

We will refer to the eigenvectors you listed above as  $\mathbf{x}_1$ ,  $\mathbf{x}_2$  and  $\mathbf{x}_3$ .

**Question 7.2** Form matrix  $P$  whose columns are the eigenvectors  $\mathbf{x}_1$ ,  $\mathbf{x}_2$ , and  $\mathbf{x}_3$  you found in the previous part. Use technology to find  $P^{-1}$ .

$$P^{-1} = \frac{1}{6} \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

**Question 7.3** Find the product  $P^{-1}AP$ . What did you get? Which of the following is true?

**Multiple Choice:**

1.  $A \sim P$
2.  $A \sim D$  for ANY diagonal matrix  $D$
3.  $A \sim D$  for the diagonal matrix  $D$  whose diagonal entries are the eigenvalues of  $A$

**Problem 8** Prove that if  $A$  and  $B$  are  $n \times n$  matrices and  $A \sim B$ , then

1.  $A^{-1} \sim B^{-1}$  (provided that the inverses exist),
2.  $A^T \sim B^T$ , and
3.  $A^k \sim B^k$  for all integers  $k \geq 1$ .

(See Theorem ??)

**Problem 9** Let  $\lambda$  be an eigenvalue of  $A$  with corresponding eigenvector  $\mathbf{x}$ . If  $B = P^{-1}AP$  is similar to  $A$ , show that  $P^{-1}\mathbf{x}$  is an eigenvector of  $B$  corresponding to  $\lambda$ .

## Text Source

The text in this section is a compilation of material from Section 7.2.1 of Ken Kuttler's *A First Course in Linear Algebra* (CC-BY) and Section 5.5 of Keith Nicholson's *Linear Algebra with Applications* (CC-BY-NC-SA).

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 362-364.

Many of the Practice Problems are Exercises from W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 298-310.