
1 Similar Matrices

Definition 1. If A and B are $n \times n$ matrices, we say that A and B are *similar*, if $B = P^{-1}AP$ for some invertible matrix P . In this case we write $A \sim B$.

Similar matrices share many properties.

Theorem 1. If A and B are $n \times n$ matrices and $A \sim B$, then

1. $\det(A) = \det(B)$,
2. $\text{rank}(A) = \text{rank}(B)$,
3. A and B have the same characteristic equations, and
4. A and B have the same eigenvalues.

Proof Let $B = P^{-1}AP$ for some invertible matrix P .

For ??, $\det B = \det(P^{-1}) \det A \det P = \det A$ because $\det(P^{-1}) = 1 / \det P$ (Theorem ??).

Similarly, for ?? $\text{rank} B = \text{rank}(P^{-1}AP) = \text{rank} A$, because multiplication by an invertible matrix cannot change the rank. To see this, note that any invertible matrix is a product of elementary matrices. Multiplying by elementary matrices is equivalent to performing elementary row (column) operations on A , which does not change the row (column) space, nor the rank. It follows that similar matrices have the same rank.

As for ??,

$$\begin{aligned}\det(B - \lambda I) &= \det\{P^{-1}AP - \lambda(P^{-1}P)\} \\ &= \det\{P^{-1}(A - \lambda I)P\} \\ &= \det(A - \lambda I),\end{aligned}$$

so A and B have the same characteristic equation. Finally, ?? implies ?? because the eigenvalues of a matrix are the roots of its characteristic polynomial. ■

Remark 1. Sharing the four properties in Theorem ?? does not guarantee that two matrices are similar. The matrices

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ have the same determinant, rank, characteristic polynomial, and eigenvalues, but they are not similar because $P^{-1}IP = I$ for any invertible matrix P .

The next theorem shows that similarity is preserved under inverses, transposes, and powers:

Theorem 2. If A and B are $n \times n$ matrices and $A \sim B$, then

1. $A^{-1} \sim B^{-1}$ (provided that the inverses exist),
2. $A^T \sim B^T$, and
3. $A^k \sim B^k$ for all integers $k \geq 1$.

Proof See Practice Problem ??. ■

Practice Problems

Problem 1 Prove that if A and B are $n \times n$ matrices and $A \sim B$, then

1. $A^{-1} \sim B^{-1}$ (provided that the inverses exist),
2. $A^T \sim B^T$, and
3. $A^k \sim B^k$ for all integers $k \geq 1$.

(See Theorem ??)

Problem 2 Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

Question 2.1 The eigenvalues of A are known to be $\lambda_1 = -3$, $\lambda_2 = 3$ and $\lambda_3 = 1$. Find the corresponding eigenspaces.

$$S_{-3} = \text{span} \left(\begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

$$S_3 = \text{span} \left(\begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

$$S_1 = \text{span} \left(\begin{bmatrix} \dots \\ \dots \\ 1 \end{bmatrix} \right)$$

We will refer to the eigenvectors you listed above as $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$.

Question 2.2 Form matrix P whose columns are the eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ you found in the previous part. Use technology to find P^{-1} .

$$P^{-1} = \frac{1}{6} \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Question 2.3 Find the product $P^{-1}AP$. What did you get? Which of the following is true?

Multiple Choice:

1. $A \sim P$
2. $A \sim D$ for ANY diagonal matrix D
3. $A \sim D$ for the diagonal matrix D whose diagonal entries are the eigenvalues of A

Problem 3 Let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. If $B = P^{-1}AP$ is similar to A , show that $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to λ .

Text Source

The text in this section is a compilation of material from Section 7.2.1 of Ken Kuttler's *A First Course in Linear Algebra* (CC-BY) and Section 5.5 of Keith Nicholson's *Linear Algebra with Applications* (CC-BY-NC-SA).

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 362-364.