
1 Diagonalizable Matrices and Multiplicity

Recall that a *diagonal matrix* D is a matrix containing a zero in every entry except those on the main diagonal. More precisely, if d_{ij} is the ij^{th} entry of a diagonal matrix D , then $d_{ij} = 0$ unless $i = j$. Such matrices look like the following.

$$D = \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$$

where $*$ is a number which might not be zero.

Diagonal matrices have some nice properties, as we demonstrate below.

Exploration 1 Let $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and let $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 10 \end{bmatrix}$. Compute MD and DM using Octave.

```
% Define M
M=[1 2 3; 4 5 6; 7 8 9]

% The function diag(vector) creates a square matrix with diagonal
%vector and the remaining entries zero.
D=diag([2 -5 10])

% The function diag() can also be used to return the diagonal of a matrix.
diag_M=diag(M)

% Understanding multiplication by a diagonal matrix
product_MD=M*D
product_DM=D*M

% Raising a diagonal matrix to a power
D5=D^5
```

[Link to code.](#)

Notice the patterns present in the product matrices.

- Each row of DM is the same as its corresponding row of M multiplied by the scalar which is the corresponding diagonal element of D .
- In the product MD , it is the columns of M that have been multiplied by the diagonal elements.
- Raising D to the 5^{th} power amounts to raising each diagonal element to the 5^{th} power.

These patterns hold in general for any diagonal matrix, and they are fundamental to understanding diagonalization and its uses. We discuss diagonalization next.

Definition 1. Let A be an $n \times n$ matrix. Then A is said to be *diagonalizable* if there exists an invertible matrix P such that

$$P^{-1}AP = D$$

where D is a diagonal matrix. In other words, a matrix A is diagonalizable if it is similar to a diagonal matrix, $A \sim D$.

If we are given a matrix A that is diagonalizable, then we can write $P^{-1}AP = D$ for some matrix P , or, equivalently,

$$AP = PD \quad (1)$$

If we pause to examine Equation (??), the work that we did in Exploration ?? can help us to understand how to find P that will diagonalize A . The product PD is formed by multiplying each column of P by a scalar which is the corresponding element on the diagonal of D . To restate this, if $\mathbf{x}_i$ is column i in our matrix P , then Equation (??) tells us that

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i, \quad (2)$$

where λ_i is the i th diagonal element of D .

Of course, Equation (??) is very familiar! We see that if we are able to diagonalize a matrix A , the columns of matrix P will be the eigenvectors of A , and the corresponding diagonal entries of D will be the corresponding eigenvalues of A . This is summed up in the following theorem.

Theorem 1. An $n \times n$ matrix A is diagonalizable if and only if there is an invertible matrix P given by

$$P = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix},$$

where the columns $\mathbf{x}_i$ are eigenvectors of A .

Moreover, if A is diagonalizable, the corresponding eigenvalues of A are the diagonal entries of the diagonal matrix D .

Proof Suppose P is given as above as an invertible matrix whose columns are eigenvectors of A . To show that A is diagonalizable, we will show

$$AP = PD,$$

which is equivalent to $P^{-1}AP = D$. We have

$$AP = \begin{bmatrix} | & | & \cdots & | \\ A\mathbf{x}_1 & A\mathbf{x}_2 & \cdots & A\mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix},$$

while

$$PD = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \lambda_1 \mathbf{x}_1 & \lambda_2 \mathbf{x}_2 & \cdots & \lambda_n \mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix}$$

We can complete this half of the proof by comparing columns, and noting that

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i \quad (3)$$

for $i = 1, \dots, n$ since the $\mathbf{x}_i$ are eigenvectors of A and the λ_i are corresponding eigenvalues of A .

Conversely, suppose A is diagonalizable so that $P^{-1}AP = D$. Let

$$P = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix}$$

where the columns are the vectors $\mathbf{x}_i$ and

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Then

$$AP = PD = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

and so

$$\begin{bmatrix} | & | & \cdots & | \\ A\mathbf{x}_1 & A\mathbf{x}_2 & \cdots & A\mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \cdots & \lambda_n\mathbf{x}_n \\ | & | & \cdots & | \end{bmatrix}$$

showing the $\mathbf{x}_i$ are eigenvectors of A and the λ_i are eigenvalues. ■

Notice that because the matrix P defined above is invertible it follows that the set of eigenvectors of A , $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$, is a basis of $\mathbb{R}^n$.

We demonstrate the concept given in the above theorem in the next example. Note that not only are the columns of the matrix P formed by eigenvectors, but P must be invertible, and therefore must consist of a linearly independent set of eigenvectors.

Example 1. Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Explanation. We will use eigenvectors of A as the columns of P , and the corresponding eigenvalues of A as the diagonal entries of D . The eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 6$. We leave these computations as exercises, as well as the computations to find a basis for each eigenspace.

One possible basis for $\mathcal{S}_2$, the eigenspace corresponding to 2, is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, while a basis for $\mathcal{S}_6$ is given by

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

We construct the matrix P by using these basis elements as columns.

$$P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

You can verify (See Practice Problem) that

$$P^{-1} = \begin{bmatrix} -1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & -1/4 \end{bmatrix}$$

Thus,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & -1/4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \end{aligned}$$

You can see that the result here is a diagonal matrix where the entries on the main diagonal are the eigenvalues of A . Notice that eigenvalues on the main diagonal *must* be in the same order as the corresponding eigenvectors in P .

Exploration 2 We will continue to use matrices from Example ?? to illustrate why diagonalizable matrices are

easier to work with. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$ and let $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$. Would it be easier to compute A^5 or D^5 if

you had to do so by hand, without a computer? Certainly D^5 is easier, due to the number of zero entries!

Raising a diagonal matrix to a power amounts to simply raising each entry to that same power, whereas computing A^5 requires many more calculations. However, we learned in Example ?? that A is similar to D . We will use this to make our computation easier as follows

$$\begin{aligned} A^5 &= (PDP^{-1})^5 \\ &= (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1}) \\ &= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)D(P^{-1}P)DP^{-1} \\ &= PD(I)D(I)D(I)D(I)DP^{-1} \\ &= PD^5P^{-1} \end{aligned}$$

With this in mind, it is not as daunting to calculate A^5 by hand.

$$A^5 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 32 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 7776 \end{bmatrix} \begin{bmatrix} -1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & -1/4 \end{bmatrix} = \begin{bmatrix} 32 & 0 & 0 \\ 1936 & 3904 & -1936 \\ -3872 & -7744 & 3904 \end{bmatrix}$$

The savings in work would certainly be more pronounced for larger matrices or for powers larger than 5.

In Exploration ??, because matrix A was diagonalizable, we were able to cut down on computations. When we chose to work with D and P instead of A we worked with the eigenvalues and eigenvectors of A . Each column of P is an eigenvector of A , and so we repeatedly made use of the following theorem (with $m = 5$).

Theorem 2. Let A be an $n \times n$ matrix and suppose $A\mathbf{x} = \lambda\mathbf{x}$. Then $A^m\mathbf{x} = \lambda^m\mathbf{x}$

Proof We prove this theorem by induction on m . Clearly $A^m\mathbf{x} = \lambda^m\mathbf{x}$ holds when $m = 1$, as that was given. For the inductive step, suppose that we know $A^{m-1}\mathbf{x} = \lambda^{m-1}\mathbf{x}$. Then

$$\begin{aligned} A^m\mathbf{x} &= (AA^{m-1})\mathbf{x} \\ &= A(A^{m-1}\mathbf{x}) \\ &= A(\lambda^{m-1}\mathbf{x}) \\ &= \lambda^{m-1}A\mathbf{x} \\ &= \lambda^{m-1}\lambda\mathbf{x} \\ &= \lambda^m\mathbf{x} \end{aligned}$$

as desired. ■

Matrix A from the Example ?? and Exploration ?? had a repeated eigenvalue of 2. The next theorem and corollary show that matrices which have *distinct* eigenvalues (where none are repeated) have desirable properties.

Theorem 3. Let A be an $n \times n$ matrix, and suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$. For each i , let $\mathbf{x}_i$ be a λ_i -eigenvector of A . Then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ is linearly independent.

Proof We prove this by induction on m , the number of vectors in the set. If $m = 1$, then $\{\mathbf{x}_1\}$ is a linearly independent set because $\mathbf{x}_1 \neq \mathbf{0}$. In general, suppose we have established that the theorem is true for some $m \geq 1$. Given eigenvectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{m+1}\}$, suppose

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_{m+1}\mathbf{x}_{m+1} = \mathbf{0}. \quad (4)$$

We must show that each $c_i = 0$. Multiply both sides of (??) on the left by A and use the fact that $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ to get

$$c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \dots + c_{m+1}\lambda_{m+1}\mathbf{x}_{m+1} = \mathbf{0}. \quad (5)$$

If we multiply (??) by λ_1 and subtract the result from (??), the first terms cancel and we obtain

$$c_2(\lambda_2 - \lambda_1)\mathbf{x}_2 + c_3(\lambda_3 - \lambda_1)\mathbf{x}_3 + \dots + c_{m+1}(\lambda_{m+1} - \lambda_1)\mathbf{x}_{m+1} = \mathbf{0}.$$

Since $\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_{m+1}$ correspond to distinct eigenvalues $\lambda_2, \lambda_3, \dots, \lambda_{m+1}$, the set $\{\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_{m+1}\}$ is linearly independent by the induction hypothesis. Hence,

$$c_2(\lambda_2 - \lambda_1) = 0, \quad c_3(\lambda_3 - \lambda_1) = 0, \quad \dots, \quad c_{m+1}(\lambda_{m+1} - \lambda_1) = 0$$

and so $c_2 = c_3 = \dots = c_{m+1} = 0$ because the λ_i are distinct. It follows that (??) becomes $c_1\mathbf{x}_1 = \mathbf{0}$, which implies that $c_1 = 0$ because $\mathbf{x}_1 \neq \mathbf{0}$, and the proof is complete. ■

The corollary that follows from this theorem gives a useful tool in determining if A is diagonalizable.

Corollary 1. Let A be an $n \times n$ matrix and suppose it has n distinct eigenvalues. Then it follows that A is diagonalizable.

Remark 1. Note that Corollary ?? is NOT an “if and only if statement”. This means that if A has repeated eigenvalues it is still sometimes possible to diagonalize A , as seen in Example ??.

Definition 2. If we are able to diagonalize A , say $A = PDP^{-1}$, we say that PDP^{-1} is an *eigenvalue decomposition* of A .

Not every matrix has an eigenvalue decomposition. Sometimes we cannot find an invertible matrix P such that $P^{-1}AP = D$. Consider the following example.

Example 2. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

If possible, find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

Explanation. We see immediately (how?) that the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 1$. To find P , the next step would be to find a basis for the corresponding eigenspace S_1 . We solve the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$. Writing this equation as an augmented matrix, we already have a matrix in row echelon form:

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

We see that the eigenvectors in S_1 are of the form

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so a basis for the eigenspace S_1 is given by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. It is easy to see that we cannot form an invertible matrix P , because any two eigenvectors will be of the form $\begin{bmatrix} t \\ 0 \end{bmatrix}$, and so the second row of P would have a row of zeros, and P could not be invertible. Hence A cannot be diagonalized.

We saw earlier in Corollary ?? that an $n \times n$ matrix with n distinct eigenvalues is diagonalizable. It turns out that there are other useful diagonalizability tests.

Recall that the *algebraic multiplicity* of an eigenvalue λ is the number of times that it occurs as a root of the characteristic polynomial.

Definition 3. The *geometric multiplicity* of an eigenvalue λ is the dimension of the corresponding eigenspace S_λ .

Consider now the following lemma.

Lemma 1. Let A be an $n \times n$ matrix, and let S_{λ_1} be the eigenspace corresponding to the eigenvalue λ_1 which has algebraic multiplicity m . Then

$$\dim(S_{\lambda_1}) \leq m$$

In other words, the geometric multiplicity of an eigenvalue is less than or equal to the algebraic multiplicity of that same eigenvalue.

Proof Let k be the geometric multiplicity of λ_1 , i.e., $k = \dim(\mathcal{S}_{\lambda_1})$. Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is a basis for the eigenspace $\mathcal{S}_{\lambda_1}$. Let P be any invertible matrix having $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ as its first k columns, say

$$P = \begin{bmatrix} | & | & & | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_k & \mathbf{x}_{k+1} & \cdots & \mathbf{x}_n \\ | & | & & | & | & & | \end{bmatrix}.$$

In block form we may write

$$P = \begin{bmatrix} B & C \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} D \\ E \end{bmatrix}$$

where B is $n \times k$, C is $n \times (n - k)$, D is $k \times n$, and E is $(n - k) \times n$. We observe $I_n = P^{-1}P = \left[\begin{array}{c|c} DB & DC \\ \hline EB & EC \end{array} \right]$. This implies

$$DB = I_k, \quad DC = O_{k \times (n-k)}, \quad EB = O_{(n-k) \times k} \quad \text{and} \quad EC = I_{n-k}$$

Therefore,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} D \\ E \end{bmatrix} A \begin{bmatrix} B & C \end{bmatrix} = \left[\begin{array}{c|c} DAB & DAC \\ \hline EAB & EAC \end{array} \right] \\ &= \left[\begin{array}{c|c} \lambda_1 DB & DAC \\ \hline \lambda_1 EB & EAC \end{array} \right] = \left[\begin{array}{c|c} \lambda_1 I_k & DAC \\ \hline O & EAC \end{array} \right] \end{aligned}$$

We finish the proof by comparing the characteristic polynomials on both sides of this equation, and making use of the fact that similar matrices have the same characteristic polynomials.

$$\det(A - \lambda I) = \det(P^{-1}AP - \lambda I) = (\lambda_1 - \lambda)^k \det(EAC)$$

We see that the characteristic polynomial of A has $(\lambda_1 - \lambda)^k$ as a factor. This tells us that algebraic multiplicity of λ_1 is at least k , proving the desired inequality. ■

This result tells us that if λ is an eigenvalue of A , then the number of linearly independent λ -eigenvectors is never more than the multiplicity of λ . We now use this fact to provide a useful diagonalizability condition.

Theorem 4. Let A be an $n \times n$ matrix A . Then A is diagonalizable if and only if for each eigenvalue λ of A , the algebraic multiplicity of λ is equal to the geometric multiplicity of λ .

Proof Suppose A is diagonalizable and let $\lambda_1, \dots, \lambda_t$ be the distinct eigenvalues of A , with algebraic multiplicities $m_1, \dots, m_t$, respectively and geometric multiplicities $k_1, \dots, k_t$, respectively. Since A is diagonalizable, Theorem ?? implies that $k_1 + \dots + k_t = n$. By applying Lemma ?? t times, we have

$$n = k_1 + \dots + k_t \leq m_1 + \dots + m_t = n,$$

which is only possible if $k_i = m_i$ for $i = 1, \dots, t$.

Conversely, if the geometric multiplicity equals the algebraic multiplicity of each eigenvalue, then obtaining a basis for each eigenspace yields n eigenvectors. Applying Theorem ??, we know that these n eigenvectors are linearly independent, so Theorem ?? implies that A is diagonalizable. ■

Practice Problems

Problems ??-??

In this exercise you will "fill in the details" of Example ??.

Problem 3 Find the eigenvalues of matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$$

Problem 4 Find a basis for each eigenspace of the matrix A .

Problem 5 Compute the inverse of

$$P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Problem 6 Compute $D = P^{-1}AP$

Problem 7 Show that computing the inverse of P is not really necessary by comparing the products PD and AP .

Problem 8 In each case, decide whether the matrix A is diagonalizable. If so, find P such that $P^{-1}AP$ is diagonal.

1. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (Diagonalizable/ Not Diagonalizable)
2. $\begin{bmatrix} 3 & 0 & 6 \\ 0 & -3 & 0 \\ 5 & 0 & 2 \end{bmatrix}$ (Diagonalizable/ Not Diagonalizable)
3. $\begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 0 \\ -1 & 0 & -3 \end{bmatrix}$ (Diagonalizable/ Not Diagonalizable)
4. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ (Diagonalizable/ Not Diagonalizable)

Problem 9 Let A denote an $n \times n$ upper triangular matrix.

1. If all the main diagonal entries of A are distinct, show that A is diagonalizable.

2. If all the main diagonal entries of A are equal, show that A is diagonalizable only if it is already diagonal. Click the arrow to see the answer. The eigenvalues of A are all equal (they are the diagonal elements), so if $P^{-1}AP = D$ is diagonal, then $D = \lambda I$. Hence $A = P^{-1}(\lambda I)P = \lambda I$.

3. Show that $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is diagonalizable but that $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

Problem 10 Let A be a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (including multiplicities). Show that:

1. $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$
2. $\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

Problem 11 1. Show that two diagonalizable matrices are similar if and only if they have the same eigenvalues with the same multiplicities.

2. If A is diagonalizable, show that $A \sim A^T$.

3. Show that $A \sim A^T$ if $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Text Source

The text in this section is a compilation of material from Section 7.2.1 of Ken Kuttler's *A First Course in Linear Algebra* (CC-BY) and Section 5.5 of Keith Nicholson's *Linear Algebra with Applications* (CC-BY-NC-SA).

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 362-364.

Many of the Practice Problems are Exercises from W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 298-310.