
1 Introduction to Linear Transformations

We start by reviewing the definition of a function.

Definition 1. Let V and W be sets. A *function* f from V into W , denoted by

$$f : V \rightarrow W$$

assigns to each element x of V , a single element $y = f(x)$ of W .

The set V is called the *domain* of f , and the set W is called the *codomain*.

If $y = f(x)$, we say that x *maps to* y , and y is the *image* of x .

The collection of images of all points of V is called the *image* of V under f , or the *image* of f . (It is also known as the *range* of f .)

In algebra and calculus you worked with functions $f : \mathbb{R} \rightarrow \mathbb{R}$ whose domain and codomain were each the set of all real numbers. In linear algebra, we call our functions *transformations*. The domain and codomain of a transformation are *vector spaces*.

Exploration 1 *In this exercise we will introduce a very special type of transformation by contrasting the effects of two transformations on vectors of $\mathbb{R}^2$. We will see that some transformations have “nice” properties, while others do not. Define T_1 and T_2 as follows:*

$$\begin{aligned} T_1 : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ T_1 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) &= \begin{bmatrix} x - y \\ x \end{bmatrix} \\ T_2 : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ T_2 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) &= \begin{bmatrix} -x + y + 1 \\ y - 2 \end{bmatrix} \end{aligned}$$

Each of these transformations takes a vector in $\mathbb{R}^2$, and maps it to another vector in $\mathbb{R}^2$. To see if you understand how these transformations are defined, see if you can determine what these transformations do to the vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

$$T_1 \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} \dots\dots \\ \dots\dots \end{bmatrix} \quad \text{and} \quad T_2 \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} \dots\dots \\ \dots\dots \end{bmatrix}.$$

Now, let's take the vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and multiply it by a scalar, say 7.

$$7 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 28 \\ 21 \end{bmatrix}$$

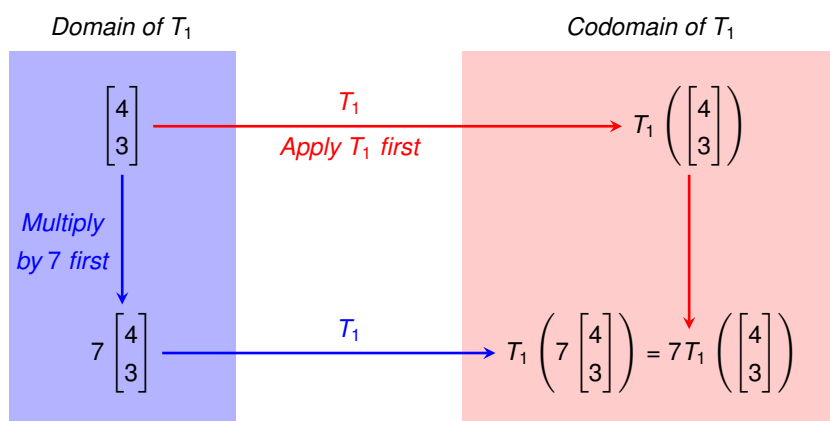
Now let's compare how T_1 and T_2 "handle" this product. Starting with T_1 , we compute:

$$T_1 \left(7 \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) = T_1 \left(\begin{bmatrix} 28 \\ 21 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ 28 \end{bmatrix}$$

Observe that multiplying the original vector by 7, then applying T_1 , has the same effect as applying T_1 to the original vector, then multiplying the image by 7. In other words,

$$T_1 \left(7 \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ 28 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = 7 T_1 \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} \right)$$

Diagrammatically, this can be represented as follows.



You should try to verify that this property does not hold for transformation T_2 . In other words,

$$T_2 \left(7 \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) \neq 7 T_2 \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} \right)$$

There is nothing special about the number 7, and it is not hard to prove that for any scalar k and vector $\mathbf{u}$ of $\mathbb{R}^2$, T_1 satisfies

$$kT_1(\mathbf{u}) = T_1(k\mathbf{u}). \quad (1)$$

It turns out that T_1 satisfies another important property. For all vectors $\mathbf{u}$ and $\mathbf{v}$ of $\mathbb{R}^2$ we have:

$$T_1(\mathbf{u} + \mathbf{v}) = T_1(\mathbf{u}) + T_1(\mathbf{v}) \quad (2)$$

We leave it to the reader to illustrate this property with a specific example (see Practice Problem ??). We will show that T_1 satisfies (??) in general.

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then

$$\begin{aligned} T_1(\mathbf{u} + \mathbf{v}) &= T_1\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T_1\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + v_1 - u_2 - v_2 \\ u_1 + v_1 \end{bmatrix} \\ &= \begin{bmatrix} u_1 - u_2 \\ u_1 \end{bmatrix} + \begin{bmatrix} v_1 - v_2 \\ v_1 \end{bmatrix} = T_1\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + T_1\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) \\ &= T_1(\mathbf{u}) + T_1(\mathbf{v}) \end{aligned}$$

It turns out that T_2 fails to satisfy this property. Can you prove that this is the case? Remember that to prove that a property DOES NOT hold, it suffices to find a counter-example. See if you can find vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$T_2(\mathbf{u} + \mathbf{v}) \neq T_2(\mathbf{u}) + T_2(\mathbf{v}). \quad (3)$$

(See Practice Problem ??.)

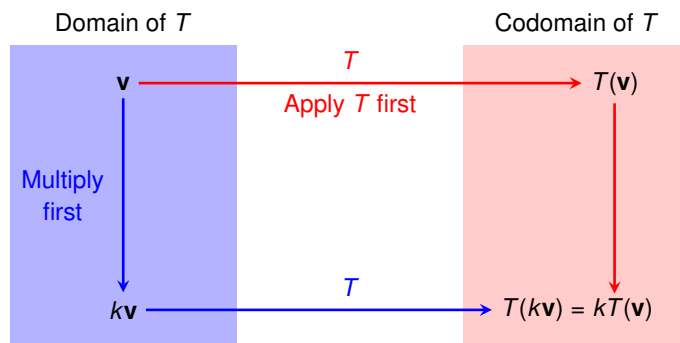
Transformations satisfying (??) and (??), like T_1 , are called *linear transformations*. Transformations like T_2 are not linear. You have already encountered several linear transformations in the form of matrix transformations in sections [Matrix Transformations](#) and [Geometric Transformations of the Plane](#).

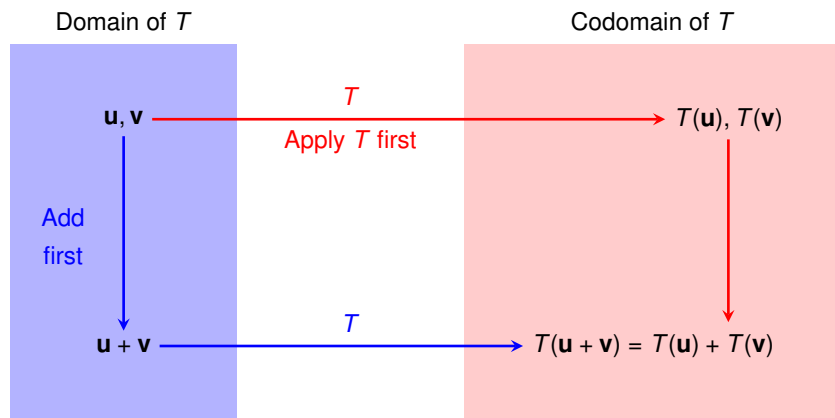
Definition 2. A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a *linear transformation* if the following are true for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, and scalars k .

$$T(k\mathbf{u}) = kT(\mathbf{u}) \quad (4)$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad (5)$$

Equations (??) and (??) of the above definition can be illustrated diagrammatically as follows.





Remark 1. Properties (??) and (??) are often combined into a single property.

$$T(k_1 \mathbf{u} + k_2 \mathbf{v}) = k_1 T(\mathbf{u}) + k_2 T(\mathbf{v}) \quad (6)$$

Example 1. Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Find each of the following:

$$1. \quad T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = T\left(2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

$$2. \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

Explanation. ?? Because T is a linear transformation, it satisfies (??). We compute:

$$\begin{aligned} T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) &= T\left(2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ &= 2T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ &= 2 \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 6 \end{bmatrix} \end{aligned}$$

?? Observe that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. By (??) we have:

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) &= T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ &= \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \end{aligned}$$

Observation 1. In Example ?? we were given the images of two vectors, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, under a linear transformation T . Based on this information, we were able to determine the images of two additional vectors: $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The reason we were able to determine $T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ is because $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ can be written as unique linear combinations of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Can every vector of $\mathbb{R}^2$ be written as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$?

Multiple Choice:

1. Yes
2. No

Is the information provided in Example ?? sufficient to determine the image of every vector in $\mathbb{R}^2$ under T ?

Multiple Choice:

1. Yes
2. No

Example 2. Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a transformation such that

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Determine whether T is a linear transformation.

Explanation. Observe that

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

If T were a linear transformation, then we would have:

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

But according to the given,

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Since $\begin{bmatrix} 6 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ we conclude that transformation T is not linear.

In Exploration ?? we introduced a transformation T_2 which turned out to be non-linear. It took some work to show that T_2 is not linear. The following theorem would have made our work easier.

Theorem 1. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then

1. $T(\mathbf{0}) = \mathbf{0}$. In other words, linear transformations map the zero vector to the zero vector.
2. T maps any line in $\mathbb{R}^n$ to a line (or the zero vector) in $\mathbb{R}^m$.

Proof To prove part ??, let $\mathbf{v}$ be any vector in $\mathbb{R}^n$. By linearity of T , we have:

$$T(\mathbf{0}) = T(\mathbf{v} - \mathbf{v}) = T(\mathbf{v}) - T(\mathbf{v}) = \mathbf{0}$$

Part ?? will become evident after we prove Theorem ?? in [Standard Matrix of a Linear Transformation](#) and combine it with Practice Problem ??. ■

Example 3. Use Theorem ?? to show that transformation T_2 of Exploration ?? is not linear.

Explanation. Recall that $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ was defined by

$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x + y + 1 \\ y - 2 \end{bmatrix}$$

We evaluate T_2 at $\mathbf{0}$:

$$T_2(\mathbf{0}) = \begin{bmatrix} -0 + 0 + 1 \\ 0 - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \neq \mathbf{0}$$

Since $T_2(\mathbf{0}) \neq \mathbf{0}$, T_2 is not linear.

Linear Transformations Induced by Matrices

Recall that a transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T(\mathbf{v}) = A\mathbf{v}$, where A is some $m \times n$ matrix, is called a matrix transformation (or transformation induced by A). As we had discovered in [Matrix Transformations](#), all matrix transformations are linear. We now formalize this result as a theorem.

Theorem 2. Let A be an $m \times n$ matrix. Define $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $T(\mathbf{v}) = A\mathbf{v}$. Then T is a linear transformation.

Proof Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$, and let k be a scalar. By properties of matrix multiplication we have:

$$T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$$

$$T(k\mathbf{u}) = A(k\mathbf{u}) = kA\mathbf{u} = kT(\mathbf{u})$$

Therefore T is a linear transformation. ■

Example 4. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation induced by

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}$$

1. Find n and m .
2. Find the image of T .

Explanation. ?? A is a 3×2 matrix, so for the expression $T(\mathbf{x}) = A\mathbf{x}$ to make sense, $\mathbf{x}$ has to be a 2×1 vector. Thus, the domain of T is $\mathbb{R}^2$ ($n = 2$). The product $A\mathbf{x}$ is a 3×1 vector. The codomain of T is $\mathbb{R}^3$ ($m = 3$).

?? By Definition ??, the image of T consists of images of all individual vectors in $\mathbb{R}^2$ under T . Every vector $\mathbf{v}$ in $\mathbb{R}^2$ can be written as $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ for some real numbers a and b . Consider the image of $\mathbf{v}$

$$T(\mathbf{v}) = T(a\mathbf{i} + b\mathbf{j}) = aT(\mathbf{i}) + bT(\mathbf{j}) = a \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$$

This shows that the range, or the image, of T consists of all linear combinations of the columns of A . In other words,

the image of T is the span of vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$. The two vectors are not scalar multiples of each other, therefore they span a plane in $\mathbb{R}^3$.

Example 5. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation induced by

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & -6 \end{bmatrix}$$

1. Find n and m .
2. Find and draw the image of T .

Explanation. ??

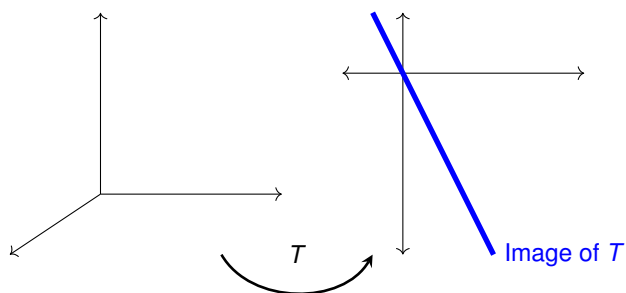
$$n = \dots, \quad m = \dots$$

?? To find the image of T , we will take a slightly different approach from what we did in Example ????.

Let $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be an arbitrary vector of $\mathbb{R}^3$. The image of $\mathbf{v}$ is given by

$$\begin{aligned} T(\mathbf{v}) &= \begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} -2 \\ 4 \end{bmatrix} + b \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c \begin{bmatrix} 3 \\ -6 \end{bmatrix} \\ &= (a(-2) + b + c(3)) \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{aligned}$$

This shows that the image of every vector in $\mathbb{R}^3$ is a scalar multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. This means that the image of T is a line in $\mathbb{R}^2$.



Linear Transformations of Subspaces of $\mathbb{R}^n$

Definition ?? defines a linear transformation as a map from $\mathbb{R}^n$ into $\mathbb{R}^m$. We will now make this definition more general by allowing the domain and the codomain of the transformation to be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$. Eventually, a linear transformation will be defined as a mapping between vector spaces.

Definition 3. Let V and W be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$. A transformation $T : V \rightarrow W$ is called a *linear transformation* if the following are true for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V , and scalars k .

$$T(k\mathbf{u}) = kT(\mathbf{u})$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

Example 6. Let V be a subspace of $\mathbb{R}^3$ consisting of all vectors in the xy -plane. Let W be a subspace of $\mathbb{R}^3$ consisting of all vectors along the z -axis. (Do a quick verification that V and W are subspaces of $\mathbb{R}^3$.) Define a

transformation $T : V \rightarrow W$ by

$$T \left(\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ a + b \end{bmatrix}$$

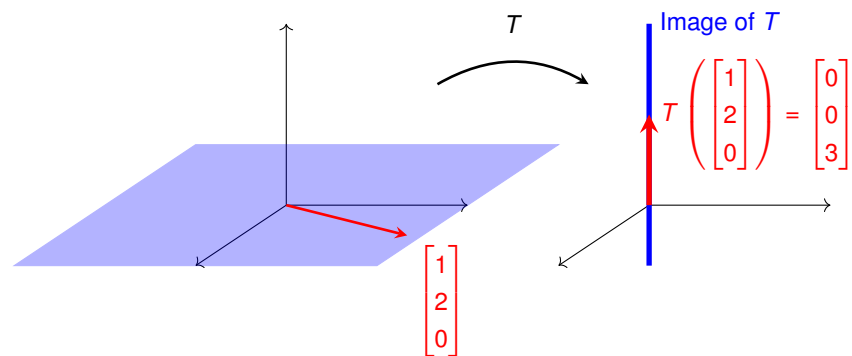
Show that T is a linear transformation, and describe its action geometrically.

Explanation. Consider two arbitrary elements $\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix}$ of V .

$$\begin{aligned} T \left(\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix} \right) &= T \left(\begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ 0 \\ (a_1 + a_2) + (b_1 + b_2) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ (a_1 + b_1) + (a_2 + b_2) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ a_1 + b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_2 + b_2 \end{bmatrix} \\ &= T \left(\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} \right) + T \left(\begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix} \right) \end{aligned}$$

Verification of the fact that $T \left(k \begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} \right) = kT \left(\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} \right)$ is similar, and we omit the details.

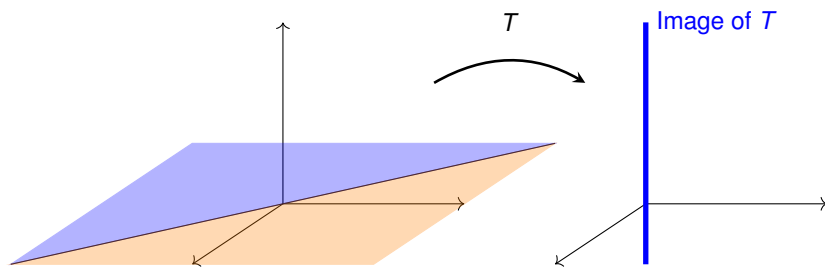
We have shown that T is a linear transformation. T maps all vectors in the xy -plane to the z -axis. The following diagram helps us visualize the action of T on a specific vector.



We can investigate further. Recall that T is defined by

$$T \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ a+b \end{bmatrix}$$

Complete each of the following statements referring to the diagram below.



The image of the line $y = -x$ under T is (a line/ a plane/ the zero vector/ all of $\mathbb{R}^3$)

The image of the orange part of the domain (the front triangle) is (the positive z-axis/ the negative z-axis/ the zero vector/ the entire z-axis)

The image of the purple part of the domain (the back triangle) is (the positive z-axis/ the negative z-axis/ the zero vector/ the entire z-axis)

We conclude this section by introducing two simple but important transformations.

Definition 4. The *identity transformation* on V , denoted by id_V , is a transformation that maps each element of V to itself.

In other words,

$$\text{id}_V : V \rightarrow V$$

is a transformation such that

$$\text{id}_V(\mathbf{v}) = \mathbf{v} \quad \text{for all } \mathbf{v} \in V$$

Definition 5. The *zero transformation*, Z , maps every element of the domain to the zero vector.

In other words,

$$Z : V \rightarrow W$$

is a transformation such that

$$Z(\mathbf{v}) = \mathbf{0} \quad \text{for all } \mathbf{v} \in V$$

Theorem 3. The identity transformation is linear.

Proof Left to the reader. (See Practice Problem ??) ■

Theorem 4. The zero transformation is linear.

Proof Left to the reader. (See Practice Problem ??) ■

Practice Problems

Problem 2 Define T_1 as follows

$$T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$
$$T_1 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x - y \\ x \end{bmatrix}$$

(See Exploration ??) Use vectors $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ to illustrate that $T_1(\mathbf{u} + \mathbf{v}) = T_1(\mathbf{u}) + T_1(\mathbf{v})$.

Problem 3 Define T_2 as follows

$$T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$
$$T_2 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} -x + y + 1 \\ y - 2 \end{bmatrix}$$

(See Exploration ??)

Use a counter-example to prove that $T_2(\mathbf{u} + \mathbf{v}) \neq T_2(\mathbf{u}) + T_2(\mathbf{v})$

Problem 4 Suppose $T : \mathbb{R}^{10} \rightarrow \mathbb{R}^2$ is a linear transformation such that $T(\mathbf{u}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $T(\mathbf{v}) = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$. Find the image of $3\mathbf{u} - \mathbf{v}$.

$$T(3\mathbf{u} - \mathbf{v}) = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

Problem 5 Let $\mathbf{u}$ be a fixed non-zero vector. Define $T_{\mathbf{u}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, by $T_{\mathbf{u}}(\mathbf{x}) = \mathbf{u} - \mathbf{x}$.

1. Describe the effect of this transformation by sketching $\mathbf{x}$ and $T_{\mathbf{u}}(\mathbf{x})$ for at least four vectors $\mathbf{x}$ and a fixed vector $\mathbf{u}$ of your choice.
2. Is $T_{\mathbf{u}}$ a linear transformation?

Problem 6 Define $P_{xy} : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, by $P_{xy} \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}$. This transformation is called an orthogonal projection onto the xy -plane. Show that P_{xy} is a linear transformation.

Problem 7 Suppose a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ maps $\mathbf{i}$ to $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{j}$ to $\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$, and $\mathbf{k}$ to $\begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$. Find the

image of $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ under T .

$$T\left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 8 Prove that the identity transformation is linear. (Theorem ??)

Problem 9 Prove that the zero transformation is linear. (Theorem ??)

Problem 10 Find the domain and the codomain of the linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ induced by A ; find and draw the image of T .

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

Domain: $\mathbb{R}^n$, where $n = \dots$

Codomain: $\mathbb{R}^m$, where $m = \dots$

Problem 11 Find the domain and the codomain of the linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ induced by A ; find and draw the image of T .

$$A = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}$$

Exercise Source

Practice Problem ?? is adopted from Problem 5.1.3 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 272.