
1 Standard Matrix of a Linear Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

In [Matrix Transformations](#) and [Introduction to Linear Transformations](#) we learned several important properties of matrix transformations of $\mathbb{R}^n$ and subspaces of $\mathbb{R}^n$. Let's summarize the main points.

Summary 1. For a matrix transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, induced by an $m \times n$ matrix A we have the following results:

- T is linear. (Theorem ??) This means that

$$T(k_1 \mathbf{u} + k_2 \mathbf{v}) = k_1 T(\mathbf{u}) + k_2 T(\mathbf{v})$$

for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and scalars k_1 and k_2 in $\mathbb{R}$.

- Columns of A are the images of the standard unit vectors of $\mathbb{R}^n$ under T . (Observation ??)

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix}$$

- The action of T on all of the elements of $\mathbb{R}^n$ is completely determined by where T maps the standard unit vectors. (See Examples ?? and ??)

The last point in the summary is so important that it is worth illustrating again.

Example 1. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation. Suppose that the only information we have about this transformation is that $T(\mathbf{i}) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $T(\mathbf{j}) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ and $T(\mathbf{k}) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Is this information sufficient to determine the

image of $\mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix}$?

Explanation. Observe that

$$\mathbf{w} = \mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$$

We find $T(\mathbf{w})$ by using the fact that T is linear.

$$T(\mathbf{w}) = T(\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = T(\mathbf{i}) - 3T(\mathbf{j}) + 6T(\mathbf{k}) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 4 \end{bmatrix} + 6 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ -7 \end{bmatrix}$$

Because of properties of linear transformations, the information about the images of the standard unit vectors proved to be sufficient for us to determine the image of $\mathbf{w}$.

In Example ??, there was nothing special about the vector $\mathbf{w}$. Any vector $\mathbf{x}$ of $\mathbb{R}^n$ can be written as a unique linear combination of the standard unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$. Therefore, the image of any vector $\mathbf{x}$ under a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is uniquely determined by the images of $\mathbf{e}_1, \dots, \mathbf{e}_n$. Knowing $T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)$ allows us to construct a matrix A , with $T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)$ as columns, that induces transformation T . We formalize this idea in a theorem.

Theorem 1. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation with

$$A = \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix}$$

as a matrix that induces T .

Proof Observe that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \dots + x_n \mathbf{e}_n$$

Because T is linear, we have

$$\begin{aligned} T(\mathbf{x}) &= T(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \dots + x_n \mathbf{e}_n) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \dots + x_n T(\mathbf{e}_n) \\ &= \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x} \end{aligned}$$

Thus, for every $\mathbf{x}$ in $\mathbb{R}^n$, we have $T(\mathbf{x}) = A\mathbf{x}$. ■

Theorem ?? shows that every matrix transformation is linear. Theorem ?? states that every linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ is a matrix transformation. We combine these results in a corollary.

Corollary 1. A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

The results of this section rely on the fact that every vector of $\mathbb{R}^n$ can be written as a unique linear combination of the standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. These vectors form the *standard basis* for $\mathbb{R}^n$. We will see in [Matrix of a Linear Transformation with Respect to Arbitrary Bases](#) that the matrix used to represent a linear transformation depends on a choice of basis. Because we are using the standard basis, it is natural to name the matrix in Theorem ?? accordingly.

Definition 1. The matrix in Theorem ?? is known as the *standard matrix of the linear transformation* T .

Example 2. The standard matrix of a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $T(\mathbf{i}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $T(\mathbf{j}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and

$T(\mathbf{k}) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ is

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 4 \end{bmatrix}$$

Example 3. Find the standard matrix of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T(\mathbf{i}) = 2\mathbf{i}$ and $T(\mathbf{j}) = 2\mathbf{j}$.

Explanation. We use the images of $\mathbf{i}$ and $\mathbf{j}$ as columns of the matrix. The standard matrix of T is

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Example 4. Find the standard matrix of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ if $T \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 1 \\ 13 \\ -1 \end{bmatrix}$ and $T \left(\begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) =$

$$\begin{bmatrix} -2 \\ 0 \\ -8 \\ 2 \end{bmatrix}.$$

Explanation. In this example we are not given the images of the standard basis vectors $\mathbf{i}$ and $\mathbf{j}$. However, we can find the images of $\mathbf{i}$ and $\mathbf{j}$ by expressing $\mathbf{i}$ and $\mathbf{j}$ as linear combinations of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$, then apply the fact that T is linear.

Let's start with the easy one.

$$\mathbf{i} = -\frac{1}{2} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Therefore, by linearity of T , we have:

$$T(\mathbf{i}) = T \left(-\frac{1}{2} \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) = -\frac{1}{2} T \left(\begin{bmatrix} -2 \\ 0 \end{bmatrix} \right) = -\frac{1}{2} \begin{bmatrix} -2 \\ 0 \\ -8 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix}$$

This gives us the first column of the standard matrix for T .

You can solve the vector equation

$$a \begin{bmatrix} 3 \\ 1 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \mathbf{j}$$

to express $\mathbf{j}$ as a linear combination of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ as follows:

$$\mathbf{j} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

By linearity of T ,

$$\begin{aligned} T(\mathbf{j}) &= T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) + \frac{3}{2} T\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 6 \\ 1 \\ 13 \\ -1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} -2 \\ 0 \\ -8 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

This gives us the second column of the standard matrix. Putting all of the information together, we get the following standard matrix for T :

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 4 & 1 \\ -1 & 2 \end{bmatrix}$$

Practice Problems

Problem 1 Suppose that a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is such that $T(\mathbf{i}) = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$ and $T(\mathbf{j}) = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$. Find

$$T\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right).$$

$$T\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 2 Suppose that a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is such that $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) =$

$$\begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}. \text{ Find the standard matrix } A \text{ of } T.$$

$$A = \begin{bmatrix} \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Problem 3 Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if T doubles the x component of every vector and triples the y component.

$$A = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Problem 4 Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if T reverses the direction of each vector.

$$A = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Problem 5 Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if T doubles the length of each vector.

$$A = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Problem 6 Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if T projects each vector onto the x -axis. (e.g. $T \left(\begin{bmatrix} 4 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$)

$$A = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Problem 7 Find the standard matrix A of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if T projects each vector onto the y -axis. (e.g. $T \left(\begin{bmatrix} 4 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$)

$$A = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$