
1 Image and Kernel of a Linear Transformation

Note to Student: In this section we will often use U , V and W to denote subspaces of $\mathbb{R}^n$, or any other finite-dimensional vector space, such as those we study in [Vector Spaces](#).

The Image of a Linear Transformation

Definition 1. Let V and W be vector spaces, and let $T : V \rightarrow W$ be a linear transformation. The *image* of T , denoted by $\text{im}(T)$, is the set

$$\text{im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$$

In other words, the image of T consists of individual images of all vectors of V .

Example 1. Consider the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

1. Find $\text{im}(T)$.
2. Illustrate the action of T with a sketch.

Explanation. ?? Let $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ 4 \end{bmatrix} + c \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Thus, every element of the image can be written as a linear combination of the columns of A . We conclude that

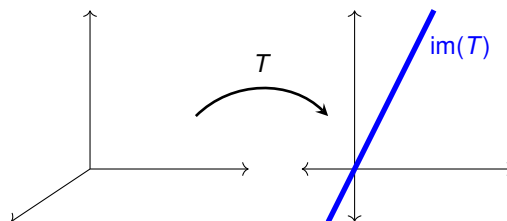
$$\text{im}(T) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right) = \text{col}(A)$$

Every column of A is a scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus,

$$\text{im}(T) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

The image of T is a line in $\mathbb{R}^2$ determined by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

?? The action of T can be illustrated with a sketch.



In Example ?? we observed that the image of the linear transformation was equal to the column space of its standard matrix. In general, it is easy to see that if $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation with standard matrix A then the following relationship holds:

$$\text{im}(T) = \text{col}(A)$$

In addition, by Theorem ??, we know that

$$\dim(\text{im}(T)) = \dim(\text{col}(A)) = \text{rank}(A)$$

Example 2. Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 0 \\ -1 & 3 & 1 & 0 & -1 \\ 3 & 0 & 0 & 3 & 6 \\ 1 & -1 & 1 & -2 & -1 \end{bmatrix}$$

Find $\text{im}(T)$ and $\dim(\text{im}(T))$.

Explanation. As in Example ??, the image of T is given by

$$\text{im}(T) = \text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 6 \\ -1 \end{bmatrix} \right) = \text{col}(A)$$

This time it is harder to detect the vectors that can be eliminated from the spanning set without affecting the span. We have to rely on the reduced row-echelon form of A .

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 0 \\ -1 & 3 & 1 & 0 & -1 \\ 3 & 0 & 0 & 3 & 6 \\ 1 & -1 & 1 & -2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see that $\text{rank}(A) = 3$, so $\dim(\text{im}(T)) = 3$.

To identify vectors that span $\text{im}(T)$, we turn to Procedure ?. We identify the first three columns as pivot columns. These columns are linearly independent and span $\text{col}(A)$. Therefore,

$$\text{im}(T) = \text{col}(A) = \text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

By Theorem ?? and Definition ??, we know that for an $m \times n$ matrix A , $\text{col}(A)$ is a subspace of $\mathbb{R}^m$. However, when vector spaces other than $\mathbb{R}^m$ are involved, it is not yet clear that $\text{im}(T)$ is a subspace of the codomain. The following theorem resolves this issue.

Theorem 1. Let $T : V \rightarrow W$ be a linear transformation. Then $\text{im}(T)$ is a subspace of W .

Proof To show that $\text{im}(T)$ is a subspace, we need to show that $\text{im}(T)$ is closed under addition and scalar multiplication.

Suppose $\mathbf{w}_1$ and $\mathbf{w}_2$ are in $\text{im}(T)$. Then there are vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. Then

$$\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2)$$

This shows that $\mathbf{w}_1 + \mathbf{w}_2$ is in $\text{im}(T)$.

For any scalar a , we have:

$$a\mathbf{w}_1 = aT(\mathbf{v}_1) = T(a\mathbf{v}_1)$$

This shows that $a\mathbf{w}_1$ is in $\text{im}(T)$. ■

We can now define the rank of a linear transformation.

Definition 2. The *rank* of a linear transformation $T : V \rightarrow W$, is the dimension of the image of T .

$$\text{rank}(T) = \dim(\text{im}(T))$$

This definition gives us the following relationship between the rank of a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and the rank of the standard matrix A associated with it.

Formula 1.

$$\text{rank}(A) = \dim(\text{col}(A)) = \dim(\text{im}(T)) = \text{rank}(T)$$

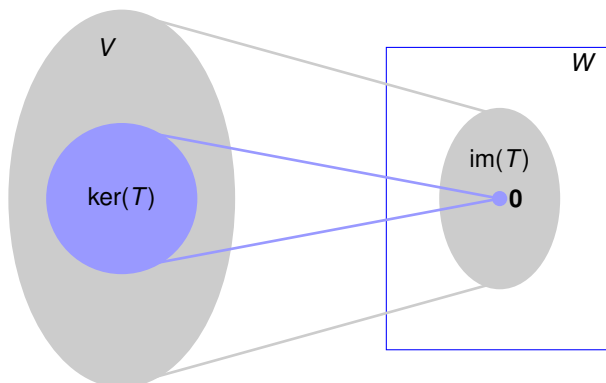
The Kernel of a Linear Transformation

Definition 3. Let V and W be vector spaces, and let $T : V \rightarrow W$ be a linear transformation. The *kernel* of T , denoted by $\ker(T)$, is the set

$$\ker(T) = \{\mathbf{v} : T(\mathbf{v}) = \mathbf{0}\}$$

In other words, the kernel of T consists of all vectors of V that map to $\mathbf{0}$ in W .

It is important to pay attention to the locations of the kernel and the image. We already proved that $\text{im}(T)$ is a subspace of the codomain. In contrast, $\ker(T)$ is located in the domain. (We will prove shortly that it is a subspace of the domain.)



Example 3. Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 0 \\ -1 & 3 & 1 & 0 & -1 \\ 3 & 0 & 0 & 3 & 6 \\ 1 & -1 & 1 & -2 & -1 \end{bmatrix}$$

1. Find $\ker(T)$
2. Is $\ker(T)$ a subspace of $\mathbb{R}^5$? If so, find $\dim(\ker(T))$.

Explanation. ?? To find the kernel of T , we need to find all vectors of $\mathbb{R}^5$ that map to $\mathbf{0}$ in $\mathbb{R}^4$. This amounts to solving the equation $A\mathbf{x} = \mathbf{0}$.

Gauss-Jordan elimination yields:

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 0 \\ -1 & 3 & 1 & 0 & -1 \\ 3 & 0 & 0 & 3 & 6 \\ 1 & -1 & 1 & -2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the kernel of T consists of all elements of the form:

$$\begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} t$$

We conclude that

$$\ker(T) = \text{span} \left(\begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

?? Since $\ker(T)$ is the span of two vectors of $\mathbb{R}^5$, we know that $\ker(T)$ is a subspace of $\mathbb{R}^5$. (See Theorem ??.) Observe that the two vectors in the spanning set are linearly independent. (How can we see this without performing computations?) Therefore $\dim(\ker(T)) = 2$.

Recall that the *null space* of a matrix A is defined to be set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. This means that if $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation with standard matrix A then

$$\ker(T) = \text{null}(A)$$

We know that $\text{null}(A)$ of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$. (See Theorem ??.) We conclude this section by showing that even when vector spaces other than $\mathbb{R}^n$ are involved, the kernel of a linear transformation is a subspace of the domain of the transformation.

Theorem 2. Let $T : V \rightarrow W$ be a linear transformation, then $\ker(T)$ is a subspace of V .

Proof To show that $\ker(T)$ is a subspace, we need to show that $\ker(T)$ is closed under addition and scalar multiplication.

Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\ker(T)$. Then,

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

This shows that $\mathbf{v}_1 + \mathbf{v}_2$ is in $\ker(T)$.

For any scalar a we have:

$$T(a\mathbf{v}_1) = aT(\mathbf{v}_1) = a\mathbf{0} = \mathbf{0}$$

This shows that $a\mathbf{v}_1$ is in $\ker(T)$. ■

Definition 4. The *nullity* of a linear transformation $T : V \rightarrow W$, is the dimension of the kernel of T .

$$\text{nullity}(T) = \dim(\ker(T))$$

This definition gives us the following relationship between nullity of a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and the nullity of the standard matrix A associated with it.

Formula 2.

$$\text{nullity}(A) = \dim(\text{null}(A)) = \dim(\ker(T)) = \text{nullity}(T)$$

Rank-Nullity Theorem for Linear Transformations

In Examples ?? and ??, we found the image and the kernel of the linear transformation $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 0 \\ -1 & 3 & 1 & 0 & -1 \\ 3 & 0 & 0 & 3 & 6 \\ 1 & -1 & 1 & -2 & -1 \end{bmatrix}$$

We also found that

$$\text{rank}(T) = \dim(\text{im}(T)) = \dim(\text{col}(A)) = \text{rank}(A) = 3$$

and

$$\text{nullity}(T) = \dim(\ker(T)) = \dim(\text{null}(A)) = \text{nullity}(A) = 2$$

Because of Rank-Nullity Theorem for matrices (Theorem ??), it is not surprising that

$$\text{rank}(T) + \text{nullity}(T) = 3 + 2 = 5 = \dim(\mathbb{R}^5)$$

The following theorem is a generalization of this result.

Theorem 3. Let $T : V \rightarrow W$ be a linear transformation. Suppose $\dim(V) = n$, then

$$\text{rank}(T) + \text{nullity}(T) = n$$

Proof By Theorem ??, $\text{im}(T)$ is a subspace of W . There exists a basis for $\text{im}(T)$ of the form $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_r)\}$. By Theorem ??, $\ker(T)$ is a subspace of V . Let $\{\mathbf{u}_1, \dots, \mathbf{u}_s\}$ be a basis for $\ker(T)$.

We will show that $\{\mathbf{u}_1, \dots, \mathbf{u}_s, \mathbf{v}_1, \dots, \mathbf{v}_r\}$ is a basis for V .

For any vector $\mathbf{v}$ in V , we have:

$$T(\mathbf{v}) = c_1 T(\mathbf{v}_1) + \dots + c_r T(\mathbf{v}_r)$$

for some scalars c_i ($1 \leq i \leq r$). Thus,

$$T(\mathbf{v}) - (c_1 T(\mathbf{v}_1) + \dots + c_r T(\mathbf{v}_r)) = \mathbf{0}$$

By linearity,

$$T(\mathbf{v} - (c_1 \mathbf{v}_1 + \dots + c_r \mathbf{v}_r)) = \mathbf{0}$$

Therefore $\mathbf{v} - (c_1 \mathbf{v}_1 + \dots + c_r \mathbf{v}_r)$ is in $\ker(T)$.

Hence there are scalars a_i ($1 \leq i \leq s$) such that

$$\mathbf{v} - (c_1 \mathbf{v}_1 + \dots + c_r \mathbf{v}_r) = a_1 \mathbf{u}_1 + \dots + a_s \mathbf{u}_s$$

Thus,

$$\mathbf{v} = (c_1 \mathbf{v}_1 + \dots + c_r \mathbf{v}_r) + (a_1 \mathbf{u}_1 + \dots + a_s \mathbf{u}_s)$$

We conclude that

$$V = \text{span}(\mathbf{u}_1, \dots, \mathbf{u}_s, \mathbf{v}_1, \dots, \mathbf{v}_r)$$

Now we need to show that $\{\mathbf{u}_1, \dots, \mathbf{u}_s, \mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

Suppose

$$c_1 \mathbf{v}_1 + \dots + c_r \mathbf{v}_r + a_1 \mathbf{u}_1 + \dots + a_s \mathbf{u}_s = \mathbf{0} \tag{1}$$

Applying T to both sides, we get

$$T(c_1 \mathbf{v}_1 + \dots + c_r \mathbf{v}_r + a_1 \mathbf{u}_1 + \dots + a_s \mathbf{u}_s) = T(\mathbf{0})$$

$$c_1 T(\mathbf{v}_1) + \dots + c_r T(\mathbf{v}_r) + a_1 T(\mathbf{u}_1) + \dots + a_s T(\mathbf{u}_s) = \mathbf{0}$$

But $T(\mathbf{u}_i) = \mathbf{0}$ for $1 \leq i \leq s$, thus

$$c_1 T(\mathbf{v}_1) + \dots + c_r T(\mathbf{v}_r) = \mathbf{0}$$

Since $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_r)\}$ is linearly independent, it follows that each $c_i = 0$.

But then Equation (??) implies that $a_1 \mathbf{u}_1 + \dots + a_s \mathbf{u}_s = \mathbf{0}$. Because $\{\mathbf{u}_1, \dots, \mathbf{u}_s\}$ is linearly independent, it follows that each $a_i = 0$.

We conclude that $\{\mathbf{u}_1, \dots, \mathbf{u}_s, \mathbf{v}_1, \dots, \mathbf{v}_r\}$ is a basis for V . Thus,

$$\dim(\ker(T)) + \dim(\operatorname{im}(T)) = s + r = n$$

■

Practice Problems

Problem 1 Describe the image and find the rank of the linear transformation $T : \mathbb{R}^5 \rightarrow \mathbb{R}^2$ induced by $A = \begin{bmatrix} 3 & 2 & 4 & 7 & 1 \\ -1 & -9 & 7 & 6 & 8 \end{bmatrix}$.

Multiple Choice:

1. $\operatorname{im}(T) = \mathbb{R}^2$
2. $\operatorname{im}(T)$ is a line in $\mathbb{R}^2$
3. $\operatorname{im}(T) = \{\mathbf{0}\}$
4. $\operatorname{im}(T) = \mathbb{R}^5$
5. $\operatorname{im}(T)$ is a plane in $\mathbb{R}^5$

$\operatorname{rank}(T) = \dots$

Problem 2 Describe the image and find the rank of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ induced by $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

Multiple Choice:

1. $\operatorname{im}(T) = \mathbb{R}^3$
2. $\operatorname{im}(T)$ is a line in $\mathbb{R}^2$
3. $\operatorname{im}(T)$ is a line in $\mathbb{R}^3$
4. $\operatorname{im}(T) = \{\mathbf{0}\}$
5. $\operatorname{im}(T)$ is a plane in $\mathbb{R}^3$

$\text{rank}(T) = \dots$

Problem 3 Suppose linear transformations $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are such that $\text{im}(T) = \text{im}(S) = \text{span} \left(\begin{bmatrix} 1 \\ -3 \end{bmatrix} \right)$. Does this mean that T and S are the same transformation? Justify your claim.

Problem 4 Describe the kernel and find the nullity of the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ induced by $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & -3 \end{bmatrix}$.

Multiple Choice:

1. $\ker(T) = \mathbb{R}^3$
2. $\ker(T) = \{\mathbf{0}\}$
3. $\ker(T) = \mathbb{R}^2$
4. $\ker(T)$ is a plane in $\mathbb{R}^3$
5. $\ker(T)$ is a line in $\mathbb{R}^3$

$\text{nullity}(T) = \dots$

Problem 5 Describe the kernel and find the nullity of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ induced by $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$.

Multiple Choice:

1. $\ker(T) = \mathbb{R}^2$
2. $\ker(T) = \{\mathbf{0}\}$
3. $\ker(T)$ is a line in $\mathbb{R}^2$

$\text{nullity}(T) = \dots$

Problem 6 Describe the kernel and find the nullity of the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^5$ induced by $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

Multiple Choice:

1. $\ker(T)$ is a plane in $\mathbb{R}^3$

2. $\ker(T)$ is a line in $\mathbb{R}^3$

3. $\ker(T)$ is a line in $\mathbb{R}^5$

4. $\ker(T) = \mathbb{R}^3$

5. $\ker(T) = \{\mathbf{0}\}$

$\text{nullity}(T) = \dots$

Problem 7 Suppose a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is such that $\text{im}(T)$ is a plane in $\mathbb{R}^3$. Then

$\text{rank}(T) = \dots$

$\text{nullity}(T) = \dots$

Problem 8 Suppose a linear transformation $T : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ is such that $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ in $\mathbb{R}^5$. Then

$\text{rank}(T) = \dots$

$\text{nullity}(T) = \dots$

Problem 9 Let $T : \mathbb{R}^6 \rightarrow \mathbb{R}^4$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 2 & -1 & 1 & -2 & 1 & 1 \\ 1 & 2 & 3 & 6 & -4 & 1 \\ 0 & 2 & 2 & 4 & -2 & -1 \\ 1 & 3 & 2 & 6 & -3 & 2 \end{bmatrix}$$

Find $\text{im}(T)$ and $\ker(T)$ if the reduced row-echelon form of A is

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 10 Let $V = \text{span} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$, and let $T : V \rightarrow \mathbb{R}^2$ be a linear transformation defined by $T(\mathbf{v}) = 2\mathbf{v}$. Find $\text{im}(T)$ and $\ker(T)$.

Problem 11 Suppose a linear transformation T is induced by a 4×6 matrix A . Let S be a linear transformation induced by A^T . Find $\text{nullity}(S)$, if $\text{nullity}(T) = 3$. Prove your claim.