
1 Matrices of Linear Transformations with Respect to Arbitrary Bases

We know that every linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ is a matrix transformation (Theorem ?? of [Standard Matrix of a Linear Transformation from \$\mathbb{R}^n\$ to \$\mathbb{R}^m\$](#)). What about linear transformations between vector spaces other than $\mathbb{R}^n$? In this section we will learn to represent linear transformations between arbitrary finite-dimensional vector spaces using matrices. To do so, we will use the fact that every n -dimensional vector space is isomorphic to $\mathbb{R}^n$ (Corollary ?? of [Isomorphic Vector Spaces](#)). What we do here will serve as yet another example of how isomorphisms can be used to translate problems in one vector space to another, more convenient, vector space.

Exploration 1 In [Isomorphic Vector Spaces](#) we introduced the transformation

$$\tau_2 : \mathbb{M}_{2,2} \rightarrow \mathbb{P}^3$$

defined by

$$\tau_2 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a + (a+c)x + (b-c)x^2 + dx^3$$

You should verify that τ_2 is linear. (See Practice Problem ??.)

We will examine τ_2 in an effort to find a way to represent it with a matrix. (In the process, we will also end up proving that τ_2 is an isomorphism, which is what you were challenged to do in [Isomorphic Vector Spaces](#).)

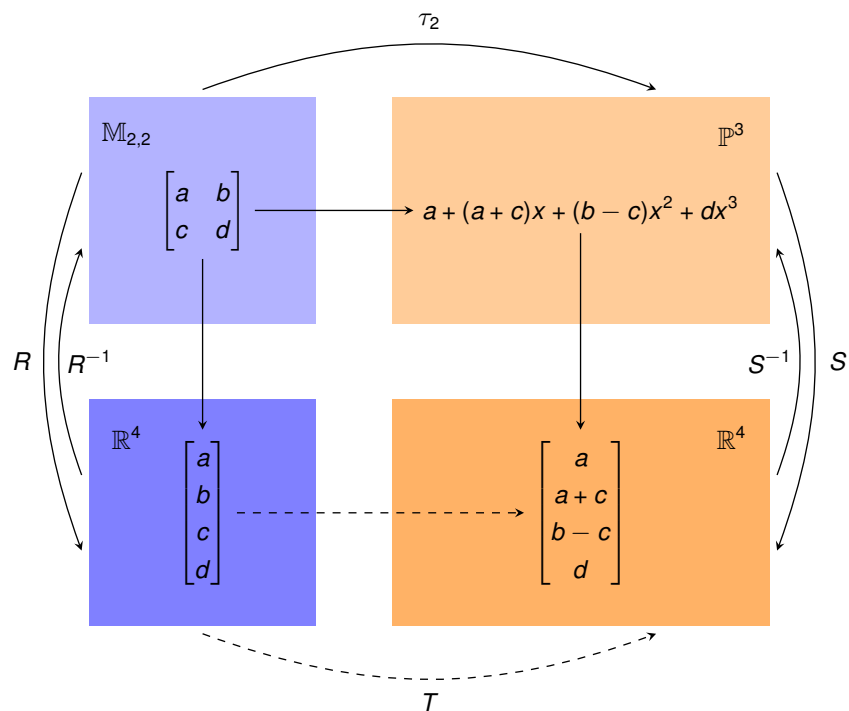
We will start by selecting a basis for each of $\mathbb{M}_{2,2}$ and $\mathbb{P}^3$. We can choose any basis for either space, but we will choose bases that will make computations easier.

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
$$\mathcal{C} = \{1, x, x^2, x^3\}$$

be our ordered bases or choice for $\mathbb{M}_{2,2}$ and $\mathbb{P}^3$, respectively.

Recall that a coordinate vector isomorphism maps a vector to its coordinate vector with respect to the given ordered basis (Theorem ?? of [Isomorphic Vector Spaces](#)). In the diagram below, let R and S be coordinate vector isomorphisms with respect to $\mathcal{B}$ and $\mathcal{C}$.



Define $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ by

$$T \left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = \begin{bmatrix} a \\ a+c \\ b-c \\ d \end{bmatrix}$$

Observe that $T = S \circ \tau_2 \circ R^{-1}$. Because T is a composition of linear transformations, T itself is linear (Theorem ?? of *Composition and Inverses of Linear Transformations*). Thus, we should be able to find the standard matrix for T . To do this, find the images of the standard unit vectors and use them to create the standard matrix A for T .

$$A = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

We say that A is the matrix of τ_2 with respect to ordered bases $\mathcal{B}$ and $\mathcal{C}$.

As a side-note, observe that $\tau_2 = S^{-1} \circ T \circ R$. Observe also that T is invertible because A is invertible. So, T is an isomorphism. As a composition of isomorphisms, τ_2 is an isomorphism (Theorem ?? of *Isomorphic Vector Spaces*). While we could have proved this result directly, as you were challenged to do in *Isomorphic Vector Spaces*, this approach is much less tedious.

Exploration 2 Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$
$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

In Example ?? we defined V and W as follows:

$$V = \text{span}(\mathbf{v}_1, \mathbf{v}_2) \quad \text{and} \quad W = \text{span}(\mathbf{w}_1, \mathbf{w}_2)$$

Geometrically, V and W are planes in $\mathbb{R}^3$. We chose

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\} \quad \text{and} \quad \mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2\}$$

as ordered bases of V and W , respectively.

We also defined a linear transformation $T : V \rightarrow W$ by

$$T(\mathbf{v}_1) = 2\mathbf{w}_1 - 3\mathbf{w}_2 \quad \text{and} \quad T(\mathbf{v}_2) = -\mathbf{w}_1 + 4\mathbf{w}_2$$

Our goal now is to find a matrix for T with respect to $\mathcal{B}$ and $\mathcal{C}$.

The information given in this problem is slightly different from the information in Exploration ?. Instead of being given an expression for the image of a generic vector of V , we are only given the images of the two basis vectors of V . But this information is sufficient to determine the linear transformation.

As before, we will map vectors of V and W to their coordinate vectors. Where are the coordinate vectors located?

1. $[\mathbf{v}_1]_{\mathcal{B}}$ and $[\mathbf{v}_2]_{\mathcal{B}}$ are elements of $(\mathbb{R}, / \mathbb{R}^2, / \mathbb{R}^3)$
2. $[\mathbf{w}_1]_{\mathcal{C}}$ and $[\mathbf{w}_2]_{\mathcal{C}}$ are elements of $(\mathbb{R}, / \mathbb{R}^2, / \mathbb{R}^3)$

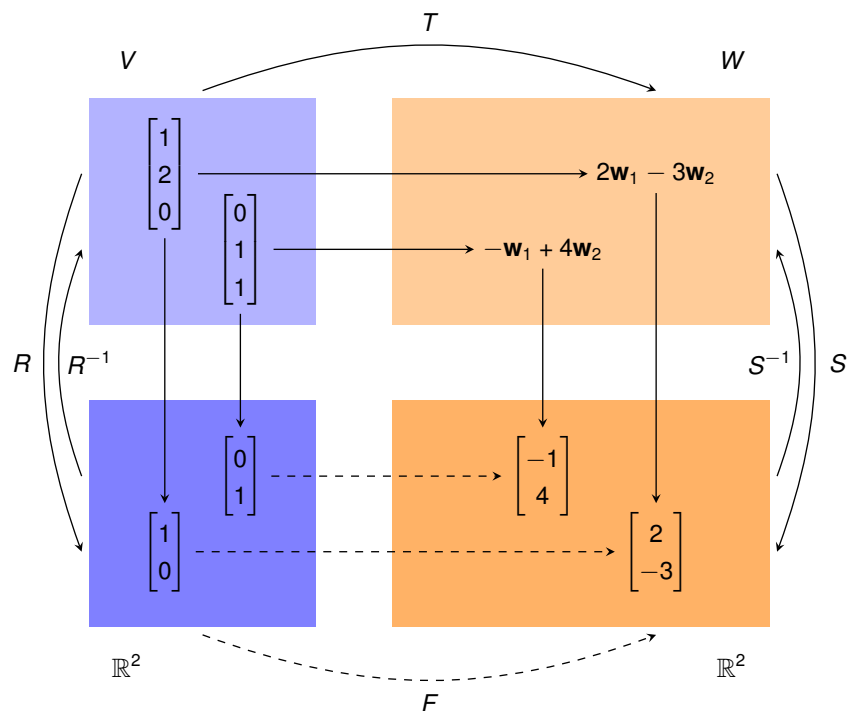
Now we find the coordinate vectors.

1.

$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} \dots \\ \dots \end{bmatrix} \quad \text{and} \quad [\mathbf{v}_2]_{\mathcal{B}} = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

2.

$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} \dots \\ \dots \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$$



Here is a diagram that summarizes this information. (Press the arrow on the right to expand.)

Define $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $F = S \circ T \circ R^{-1}$. F is a linear transformation that maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$, respectively. Thus, the standard matrix for F is:

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

We say that A is a matrix for T with respect to $\mathcal{B}$ and $\mathcal{C}$.

Let's take a look at what this matrix can do for us. Recall that in Example ?? we found that the image of $\mathbf{v} = 2\mathbf{v}_1 + \mathbf{v}_2$ is

$$T(\mathbf{v}) = T(2\mathbf{v}_1 + \mathbf{v}_2) = 2T(\mathbf{v}_1) + T(\mathbf{v}_2) = 3\mathbf{w}_1 - 2\mathbf{w}_2$$

This result can be obtained by finding the product of A with the coordinate vector of $\mathbf{v}$.

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Given any vector $\mathbf{u} = a\mathbf{v}_1 + b\mathbf{v}_2$ of V , we can find $T(\mathbf{u})$ as follows:

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a - b \\ -3a + 4b \end{bmatrix}$$

This gives us

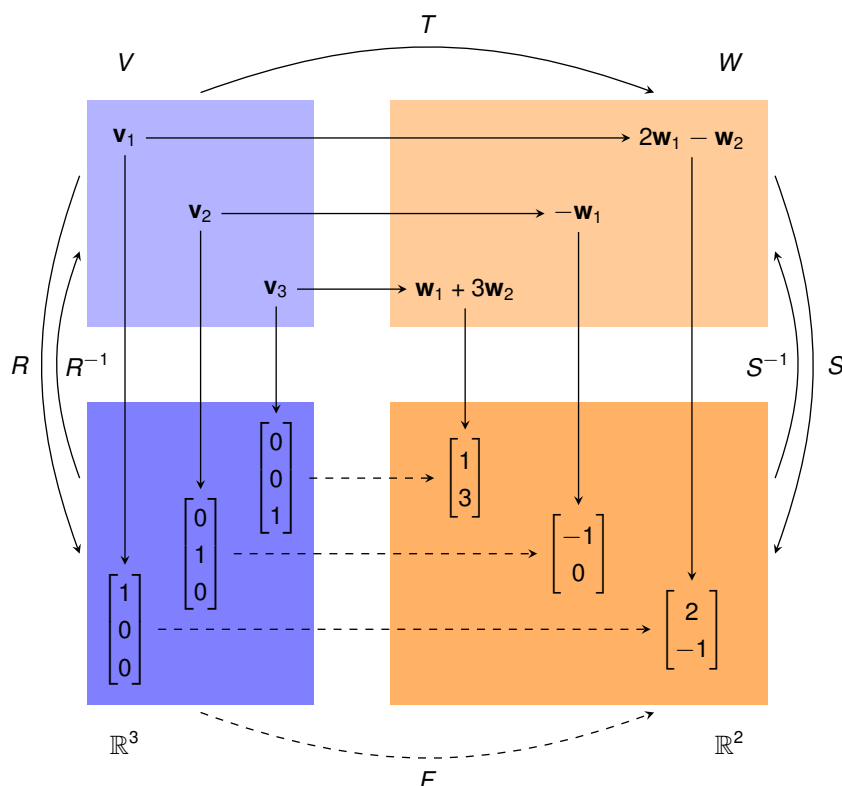
$$T(\mathbf{u}) = T(a\mathbf{v}_1 + b\mathbf{v}_2) = (2a - b)\mathbf{w}_1 + (-3a + 4b)\mathbf{w}_2$$

Example 1. Let V and W be vector spaces with ordered bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2\}$, respectively. Define a linear transformation $T : V \rightarrow W$ by

$$T(\mathbf{v}_1) = 2\mathbf{w}_1 - \mathbf{w}_2, \quad T(\mathbf{v}_2) = -\mathbf{w}_1, \quad T(\mathbf{v}_3) = \mathbf{w}_1 + 3\mathbf{w}_2$$

Find the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$, and use it to find $T(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3)$. Verify your answer by computing $T(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3)$ directly.

Explanation. We start with a diagram:



Looking at the images of the standard unit vectors in $\mathbb{R}^3$, we can construct the standard matrix A of T with respect to $\mathcal{B}$ and $\mathcal{C}$.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

Applying this matrix to the coordinate vector of $2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3$ we get

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$

This means that $T(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3) = 8\mathbf{w}_1 + 7\mathbf{w}_2$. We can verify this by direct computation as follows:

$$\begin{aligned} T(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3) &= 2T(\mathbf{v}_1) - T(\mathbf{v}_2) + 3T(\mathbf{v}_3) \\ &= 2(2\mathbf{w}_1 - \mathbf{w}_2) - (-\mathbf{w}_1) + 3(\mathbf{w}_1 + 3\mathbf{w}_2) \\ &= 8\mathbf{w}_1 + 7\mathbf{w}_2 \end{aligned}$$

The Matrix of a Linear Transformation

In this section we will formalize the process for finding the matrix of a linear transformation with respect to arbitrary bases that we established through earlier examples.

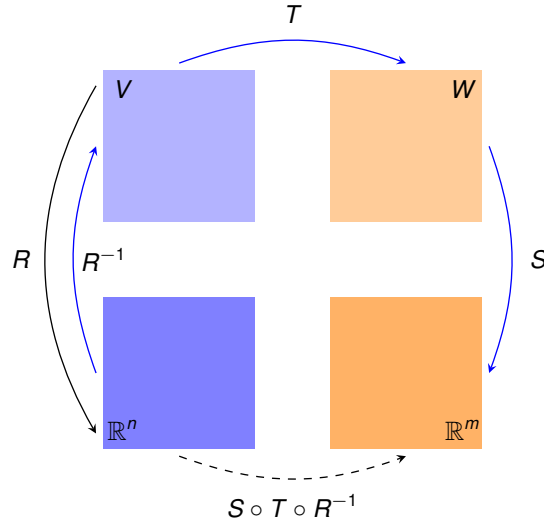
Let V and W be vector spaces with ordered bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$, respectively. Suppose $T : V \rightarrow W$ is a linear transformation. Our goal is to find a matrix for T with respect to $\mathcal{B}$ and $\mathcal{C}$.

Observe that $\dim(V) = n = \dim(\mathbb{R}^n)$ and $\dim(W) = m = \dim(\mathbb{R}^m)$. Let $R : V \rightarrow \mathbb{R}^n$ and $S : W \rightarrow \mathbb{R}^m$ be coordinate vector isomorphisms defined by

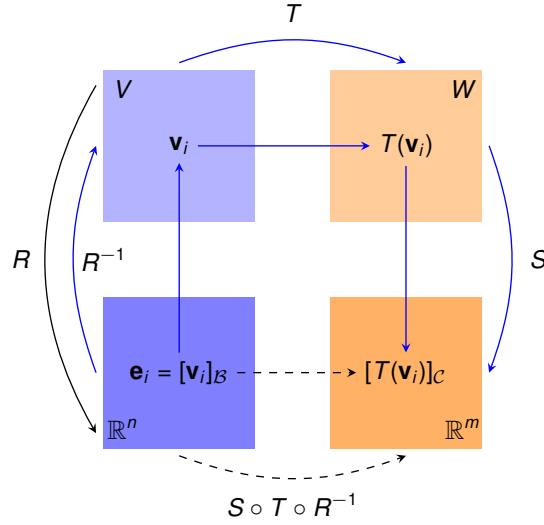
$$R(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}} \quad \text{and} \quad S(\mathbf{w}) = [\mathbf{w}]_{\mathcal{C}}$$

We know that R is an isomorphism and R^{-1} exists. Consider the transformation

$$S \circ T \circ R^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^m$$



As a composition of linear transformation, $S \circ T \circ R^{-1}$ is linear and thus has a standard matrix. To find it, we need to determine the images of standard unit vectors $\mathbf{e}_i$ under $S \circ T \circ R^{-1}$. We have the following:



Vectors $[T(\mathbf{v}_i)]_{\mathcal{C}}$ will become the columns of the standard matrix. We summarize this discussion as a theorem.

Theorem 1. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\mathcal{C}$, respectively. Suppose $T : V \rightarrow W$ is a linear transformation.

$$\text{Let } A = \begin{bmatrix} | & | & & | \\ [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \dots & [T(\mathbf{v}_n)]_{\mathcal{C}} \\ | & | & & | \end{bmatrix}$$

Then $A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$ for all vectors $\mathbf{v}$ in V .

Definition 1. Matrix A of Theorem ?? is called the matrix of T with respect to ordered bases $\mathcal{B}$ and $\mathcal{C}$.

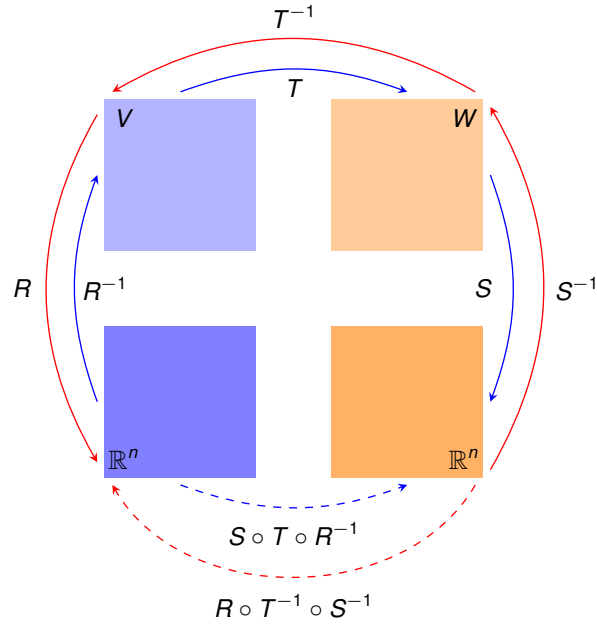
In conclusion, observe how isomorphisms helped us solve the matrix of a linear transformation problem. The coordinate mappings R and S are isomorphisms. This means that V and $\mathbb{R}^n$ are isomorphic and have the same structural properties. The same is true for W and $\mathbb{R}^m$. In this abstract discussion, we do not know anything about the elements of V and W , but isomorphisms allow us to take a problem that we do not know much about and transform it to a familiar problem involving familiar spaces.

The Inverse of a Linear Transformation and its Matrix

Let V and W be vector spaces. Suppose $T : V \rightarrow W$ is an invertible linear transformation. This, of course, means that T is an isomorphism, which means that

$$\dim(V) = \dim(W)$$

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ be ordered bases of V and W , respectively. We can find the matrix of T^{-1} with respect to $\mathcal{C}$ and $\mathcal{B}$ by finding the standard matrix of the linear transformation $R \circ T^{-1} \circ S^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$.



Observe that $R \circ T^{-1} \circ S^{-1}$ is the inverse of $S \circ T \circ R^{-1}$. So, if A is the standard matrix of $S \circ T \circ R^{-1}$, then A^{-1} is the standard matrix of $R \circ T^{-1} \circ S^{-1}$. Thus, A^{-1} is the matrix of T^{-1} with respect to $\mathcal{C}$ and $\mathcal{B}$.

Example 2. In Exploration Problem ?? of [Composition and Inverses of Linear Transformations](#), we introduced the following set up.

Let

$$V = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Define a linear transformation

$$T : V \rightarrow \mathbb{R}^2$$

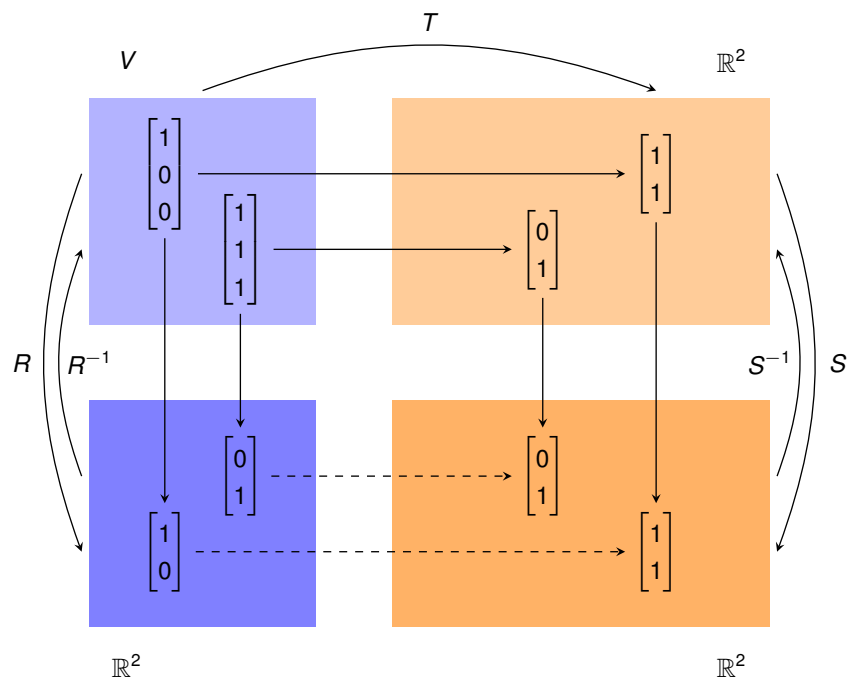
by

$$T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In Example ?? of [Existence of the Inverse of a Linear Transformation](#), we proved that T is invertible.

Find the matrix of T^{-1} with respect to ordered bases $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Explanation. Consider the diagram:



This gives us the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$:

$$A = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

We now find A^{-1} .

$$A^{-1} = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

A^{-1} is the matrix of T^{-1} with respect to $\mathcal{C}$ and $\mathcal{B}$.

Practice Problems

Problem 3 Verify that τ_2 of Exploration ?? is linear.

Problems ??-??

Define $T : \mathbb{M}_{2,2} \rightarrow \mathbb{L}$ by

$$T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + b) + (c + d)x$$

Problem 4 Show that T is linear.

Problem 5 Is T an isomorphism? (Yes, / No)

Problem 6 Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\mathcal{C} = \{1, x\}$$

be ordered bases for $\mathbb{M}_{2,2}$ and $\mathbb{L}$, respectively.

Find the matrix for T with respect to $\mathcal{B}$ and $\mathcal{C}$.

Answer:

$$\begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

Problem 7 Let V and W be vector spaces with ordered bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, respectively. Define a linear transformation $T : V \rightarrow W$ by

$$T(\mathbf{v}_1) = \mathbf{w}_1 + 2\mathbf{w}_2 - \mathbf{w}_3, \quad T(\mathbf{v}_2) = 3\mathbf{w}_2 + \mathbf{w}_3$$

Find the matrix A of T with respect to $\mathcal{B}$ and $\mathcal{C}$, and use it to find $T(-\mathbf{v}_1 - 3\mathbf{v}_2)$. Verify your answer by computing $T(-\mathbf{v}_1 - 3\mathbf{v}_2)$ directly.

$$A = \begin{bmatrix} \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{bmatrix}$$

$$T(-\mathbf{v}_1 - 3\mathbf{v}_2) = \dots \mathbf{w}_1 + \dots \mathbf{w}_2 + \dots \mathbf{w}_3$$

Problems ??-??

Let

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{w}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let V and W be subspaces of $\mathbb{R}^3$ with ordered bases

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\} \quad \text{and} \quad \mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2\}$$

respectively.

Let $T : V \rightarrow W$ be a linear transformation such that

$$T(\mathbf{v}_1) = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

Problem 8 Show that $\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ lie in W by expressing them as linear combinations of $\mathbf{w}_1$ and $\mathbf{w}_2$.

$$\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \dots \mathbf{w}_1 + \dots \mathbf{w}_2$$

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \dots \mathbf{w}_1 + \dots \mathbf{w}_2$$

Problem 9 Find the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$.

$$\begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Problem 10 Find the matrix of T^{-1} with respect to $\mathcal{C}$ and $\mathcal{B}$.

$$\begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Problem 11 Find $T^{-1} \left(\begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix} \right)$.

$$T^{-1} \left(\begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$