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# 1 Matrices of Linear Transformations with Respect to Arbitrary Bases

We know that every linear transformation from  $\mathbb{R}^n$  into  $\mathbb{R}^m$  is a matrix transformation (Theorem ??). However, the matrix that represents a linear transformation is tied to the basis we choose. So far we have only used the standard basis of  $\mathbb{R}^n$ . What about linear transformations between vector spaces other than  $\mathbb{R}^n$ ? In this section we will learn to represent linear transformations between arbitrary finite-dimensional vector spaces using matrices. To do so, we will use the fact that every  $n$ -dimensional vector space is isomorphic to  $\mathbb{R}^n$ . (Theorem ??). What we do here will serve as yet another example of how isomorphisms can be used to translate problems in one vector space to another, more convenient, vector space.

**Exploration 1** Define a transformation

$$\tau : \mathbb{M}_{2,2} \rightarrow \mathbb{P}^3$$

by

$$\tau \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a + (a + c)x + (b - c)x^2 + dx^3$$

You should verify that  $\tau$  is linear. (See Practice Problem ??.)

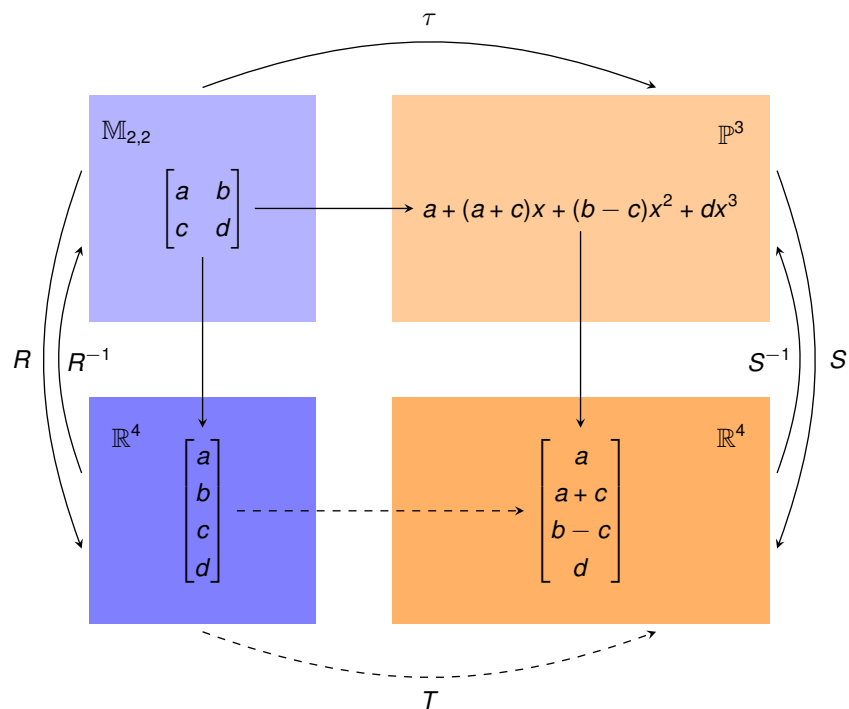
We will examine  $\tau$  in an effort to find a way to represent it with a matrix. The matrix we find will depend on our choice of bases. We will choose bases that will make computations easier.

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
$$\mathcal{C} = \{1, x, x^2, x^3\}$$

be our ordered bases of choice for  $\mathbb{M}_{2,2}$  and  $\mathbb{P}^3$ , respectively.

Recall that a coordinate isomorphism maps a vector to its coordinate vector with respect to the given ordered basis (Theorem ??). In the diagram below, let  $R$  and  $S$  be coordinate isomorphisms with respect to  $\mathcal{B}$  and  $\mathcal{C}$ , respectively.



Define  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  by

$$T \left( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = \begin{bmatrix} a \\ a+c \\ b-c \\ d \end{bmatrix}$$

Observe that  $T = S \circ \tau \circ R^{-1}$ . Because  $T$  is a composition of linear transformations,  $T$  itself is linear (Theorem ??). Thus, we should be able to find the standard matrix for  $T$ . To do this, find the images of the standard unit vectors and use them to create the standard matrix  $A$  for  $T$ .

$$A = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

We say that  $A$  is the matrix of  $\tau$  with respect to ordered bases  $\mathcal{B}$  and  $\mathcal{C}$ .

**Exploration 2** Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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Define  $V$  and  $W$  as follows:

$$V = \text{span}(\mathbf{v}_1, \mathbf{v}_2) \quad \text{and} \quad W = \text{span}(\mathbf{w}_1, \mathbf{w}_2)$$

Geometrically,  $V$  and  $W$  are planes in  $\mathbb{R}^3$ . Let's choose

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\} \quad \text{and} \quad \mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2\}$$

as ordered bases of  $V$  and  $W$ , respectively.

Define a transformation  $T : V \rightarrow W$  by

$$T(\mathbf{v}_1) = 2\mathbf{w}_1 - 3\mathbf{w}_2 \quad \text{and} \quad T(\mathbf{v}_2) = -\mathbf{w}_1 + 4\mathbf{w}_2$$

(You should mentally verify that  $T$  is linear.)

Our goal now is to find a matrix for  $T$  with respect to  $\mathcal{B}$  and  $\mathcal{C}$ .

The information given in this problem is slightly different from the information in Exploration ???. Instead of being given an expression for the image of a generic vector of  $V$ , we are only given the images of the two basis vectors of  $V$ . But this information is sufficient to determine the linear transformation.

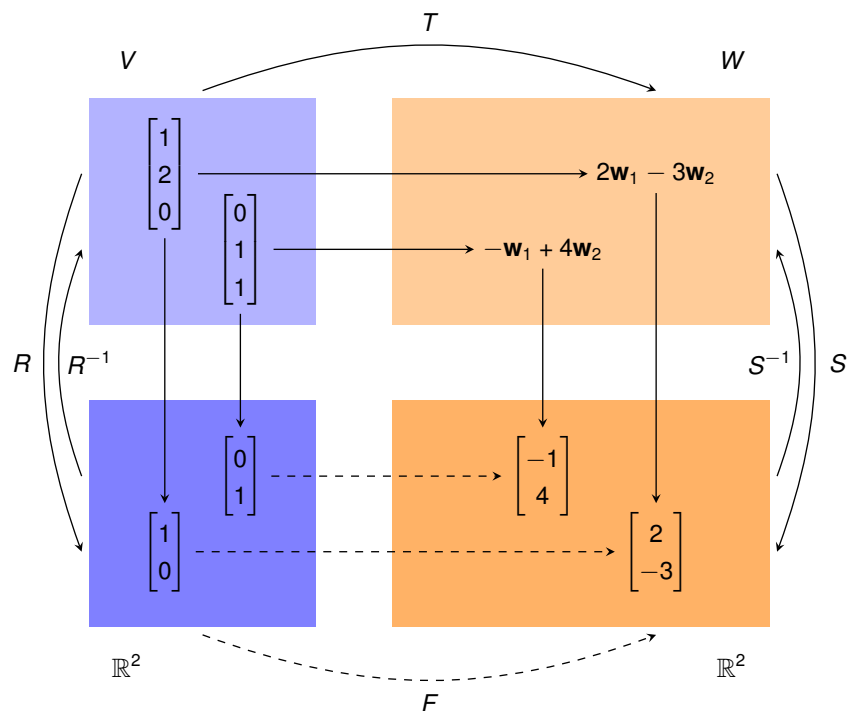
As before, we will map vectors of  $V$  and  $W$  to their coordinate vectors. Where are the coordinate vectors located?

1.  $[\mathbf{v}_1]_{\mathcal{B}}$  and  $[\mathbf{v}_2]_{\mathcal{B}}$  are elements of  $(\mathbb{R}, / \mathbb{R}^2, / \mathbb{R}^3)$
2.  $[\mathbf{w}_1]_{\mathcal{C}}$  and  $[\mathbf{w}_2]_{\mathcal{C}}$  are elements of  $(\mathbb{R}, / \mathbb{R}^2, / \mathbb{R}^3)$

Now we find the coordinate vectors.

1. 
$$[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} \cdots \cdots \\ \cdots \cdots \end{bmatrix} \quad \text{and} \quad [\mathbf{v}_2]_{\mathcal{B}} = \begin{bmatrix} \cdots \cdots \\ \cdots \cdots \end{bmatrix}$$
2. 
$$[T(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} \cdots \cdots \\ \cdots \cdots \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} \cdots \cdots \\ \cdots \cdots \end{bmatrix}$$

The diagram below summarizes this information.



Define  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $F = S \circ T \circ R^{-1}$ .  $F$  is a linear transformation that maps  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  to  $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ , respectively. Thus, the standard matrix for  $F$  is:

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

We say that  $A$  is a matrix for  $T$  with respect to  $\mathcal{B}$  and  $\mathcal{C}$ .

Let's take a look at what this matrix can do for us. Suppose we want to find the image of  $\mathbf{v} = 2\mathbf{v}_1 + \mathbf{v}_2$  under  $T$ . We can compute this directly, as follows:

$$T(\mathbf{v}) = T(2\mathbf{v}_1 + \mathbf{v}_2) = 2T(\mathbf{v}_1) + T(\mathbf{v}_2) = 3\mathbf{w}_1 - 2\mathbf{w}_2$$

This result can also be obtained by finding the product of  $A$  with the coordinate vector of  $\mathbf{v}$ .

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Given any vector  $\mathbf{u} = a\mathbf{v}_1 + b\mathbf{v}_2$  of  $V$ , we can find  $T(\mathbf{u})$  as follows:

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a - b \\ -3a + 4b \end{bmatrix}$$

This gives us

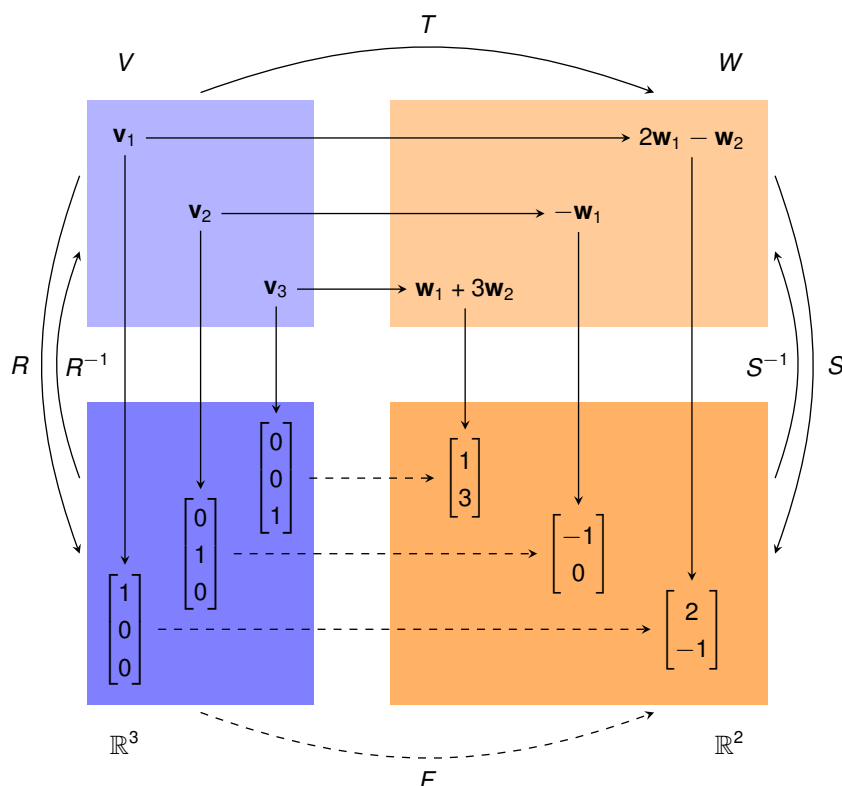
$$T(\mathbf{u}) = T(a\mathbf{v}_1 + b\mathbf{v}_2) = (2a - b)\mathbf{w}_1 + (-3a + 4b)\mathbf{w}_2$$

**Example 1.** Let  $V$  and  $W$  be vector spaces with ordered bases  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  and  $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2\}$ , respectively. Define a linear transformation  $T : V \rightarrow W$  by

$$T(\mathbf{v}_1) = 2\mathbf{w}_1 - \mathbf{w}_2, \quad T(\mathbf{v}_2) = -\mathbf{w}_1, \quad T(\mathbf{v}_3) = \mathbf{w}_1 + 3\mathbf{w}_2$$

Find the matrix of  $T$  with respect to  $\mathcal{B}$  and  $\mathcal{C}$ , and use it to find  $T(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3)$ . Verify your answer by computing  $T(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3)$  directly.

**Explanation.** We start with a diagram:



Looking at the images of the standard unit vectors in  $\mathbb{R}^3$ , we can construct the standard matrix  $A$  of  $T$  with respect to  $\mathcal{B}$  and  $\mathcal{C}$ .

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

Applying this matrix to the coordinate vector of  $2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3$  we get

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$

This means that  $T(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3) = 8\mathbf{w}_1 + 7\mathbf{w}_2$ . We can verify this by direct computation as follows:

$$\begin{aligned} T(2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3) &= 2T(\mathbf{v}_1) - T(\mathbf{v}_2) + 3T(\mathbf{v}_3) \\ &= 2(2\mathbf{w}_1 - \mathbf{w}_2) - (-\mathbf{w}_1) + 3(\mathbf{w}_1 + 3\mathbf{w}_2) \\ &= 8\mathbf{w}_1 + 7\mathbf{w}_2 \end{aligned}$$

## The Matrix of a Linear Transformation

In this section we will formalize the process for finding the matrix of a linear transformation with respect to arbitrary bases that we established through earlier examples.

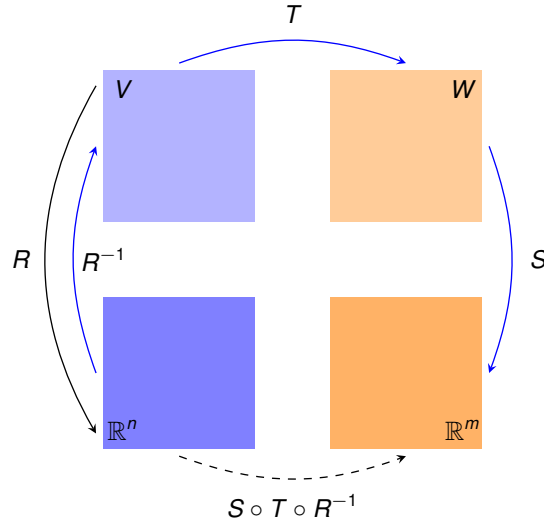
Let  $V$  and  $W$  be vector spaces with ordered bases  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  and  $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ , respectively. Suppose  $T : V \rightarrow W$  is a linear transformation. Our goal is to find a matrix for  $T$  with respect to  $\mathcal{B}$  and  $\mathcal{C}$ .

Observe that  $\dim(V) = n = \dim(\mathbb{R}^n)$  and  $\dim(W) = m = \dim(\mathbb{R}^m)$ . Let  $R : V \rightarrow \mathbb{R}^n$  and  $S : W \rightarrow \mathbb{R}^m$  be coordinate isomorphisms defined by

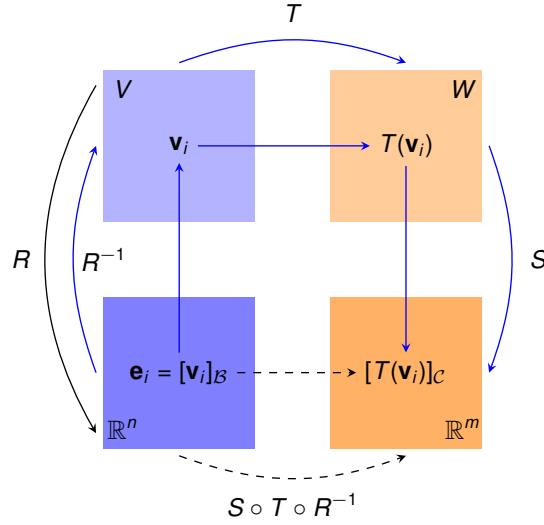
$$R(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}} \quad \text{and} \quad S(\mathbf{w}) = [\mathbf{w}]_{\mathcal{C}}$$

We know that  $R$  is an isomorphism and  $R^{-1}$  exists. Consider the transformation

$$S \circ T \circ R^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^m$$



As a composition of linear transformation,  $S \circ T \circ R^{-1}$  is linear and thus has a standard matrix. To find it, we need to determine the images of standard unit vectors  $\mathbf{e}_i$  under  $S \circ T \circ R^{-1}$ . We have the following:



Vectors  $[T(\mathbf{v}_i)]_{\mathcal{C}}$  will become the columns of the standard matrix. We summarize this discussion as a theorem.

**Theorem 1.** Let  $V$  and  $W$  be finite-dimensional vector spaces with ordered bases  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  and  $\mathcal{C}$ , respectively. Suppose  $T : V \rightarrow W$  is a linear transformation.

$$\text{Let } A = \begin{bmatrix} | & | & & | \\ [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \dots & [T(\mathbf{v}_n)]_{\mathcal{C}} \\ | & | & & | \end{bmatrix}$$

Then  $A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$  for all vectors  $\mathbf{v}$  in  $V$ .

**Definition 1.** Matrix  $A$  of Theorem ?? is called the matrix of  $T$  with respect to ordered bases  $\mathcal{B}$  and  $\mathcal{C}$ .

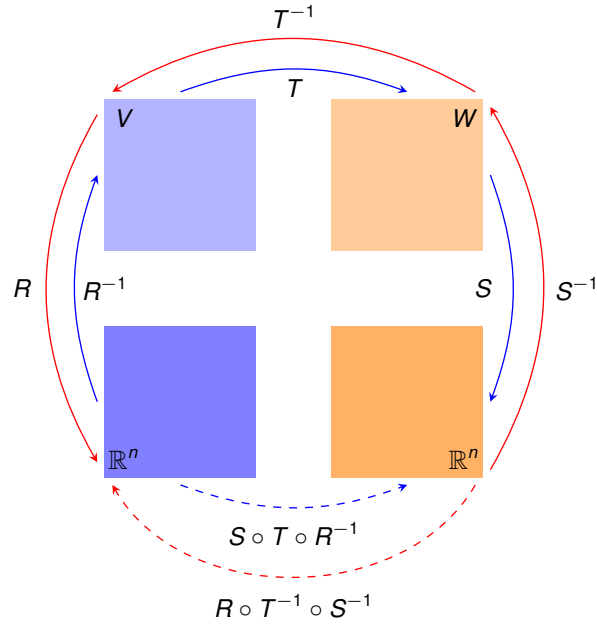
In conclusion, observe how isomorphisms helped us solve the matrix of a linear transformation problem. The coordinate mappings  $R$  and  $S$  are isomorphisms. This means that  $V$  and  $\mathbb{R}^n$  are isomorphic and have the same structural properties. The same is true for  $W$  and  $\mathbb{R}^m$ . In this abstract discussion, we do not know anything about the elements of  $V$  and  $W$ , but isomorphisms allow us to take a problem that we do not know much about and transform it to a familiar problem involving familiar spaces.

## The Inverse of a Linear Transformation and its Matrix

Let  $V$  and  $W$  be vector spaces. Suppose  $T : V \rightarrow W$  is an invertible linear transformation. This, of course, means that  $T$  is an isomorphism, which means that

$$\dim(V) = \dim(W)$$

Let  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  and  $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$  be ordered bases of  $V$  and  $W$ , respectively. We can find the matrix of  $T^{-1}$  with respect to  $\mathcal{C}$  and  $\mathcal{B}$  by finding the standard matrix of the linear transformation  $R \circ T^{-1} \circ S^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .



Observe that  $R \circ T^{-1} \circ S^{-1}$  is the inverse of  $S \circ T \circ R^{-1}$ . So, if  $A$  is the standard matrix of  $S \circ T \circ R^{-1}$ , then  $A^{-1}$  is the standard matrix of  $R \circ T^{-1} \circ S^{-1}$ . Thus,  $A^{-1}$  is the matrix of  $T^{-1}$  with respect to  $\mathcal{C}$  and  $\mathcal{B}$ .

**Example 2.** Let

$$V = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Define a linear transformation

$$T : V \rightarrow \mathbb{R}^2$$

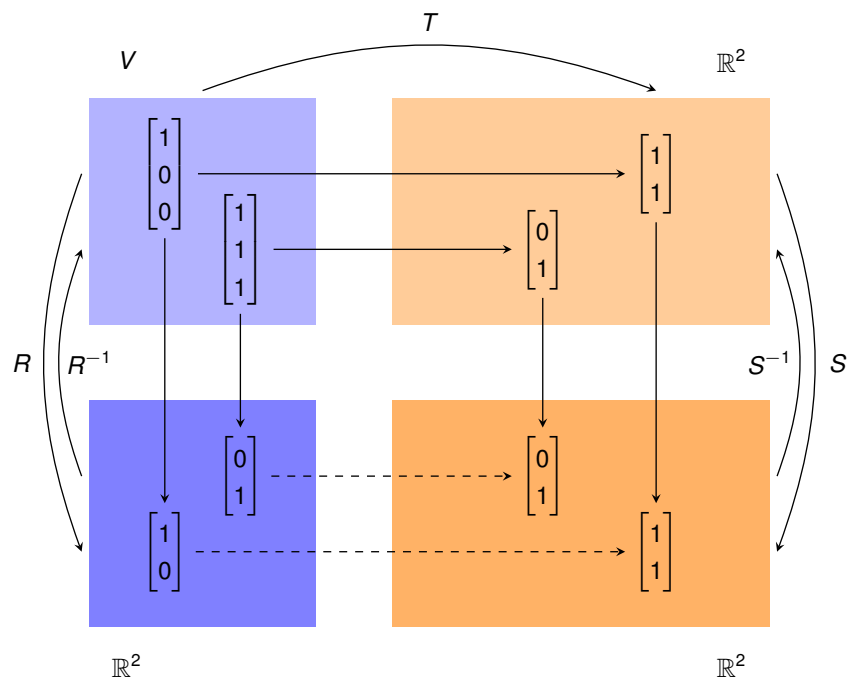
by

$$T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Conclude that  $T$  is invertible by finding the matrix of  $T^{-1}$  with respect to ordered bases  $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$  of  $\mathbb{R}^2$ , and

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ of } V.$$

**Explanation.** Consider the diagram:



This gives us the matrix of  $T$  with respect to  $\mathcal{B}$  and  $\mathcal{C}$ :

$$A = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

We now find  $A^{-1}$ .

$$A^{-1} = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

$A^{-1}$  is the matrix of  $T^{-1}$  with respect to  $\mathcal{C}$  and  $\mathcal{B}$ .

## Practice Problems

**Problem 3** Verify that  $\tau$  of Exploration ?? is linear.

**Problem 4** Define  $T : \mathbb{M}_{2,2} \rightarrow \mathbb{L}$  by

$$T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + b) + (c + d)x$$

1. Show that  $T$  is linear.

2. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\mathcal{C} = \{1, x\}$$

be ordered bases for  $\mathbb{M}_{2,2}$  and  $\mathbb{L}$ , respectively.

Find the matrix for  $T$  with respect to  $\mathcal{B}$  and  $\mathcal{C}$ .

Answer:

$$\begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

**Problem 5** Let  $V$  and  $W$  be vector spaces with ordered bases  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$  and  $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ , respectively. Define a linear transformation  $T : V \rightarrow W$  by

$$T(\mathbf{v}_1) = \mathbf{w}_1 + 2\mathbf{w}_2 - \mathbf{w}_3, \quad T(\mathbf{v}_2) = 3\mathbf{w}_2 + \mathbf{w}_3$$

Find the matrix  $A$  of  $T$  with respect to  $\mathcal{B}$  and  $\mathcal{C}$ , and use it to find  $T(-\mathbf{v}_1 - 3\mathbf{v}_2)$ . Verify your answer by computing  $T(-\mathbf{v}_1 - 3\mathbf{v}_2)$  directly.

$$A = \begin{bmatrix} \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{bmatrix}$$

$$T(-\mathbf{v}_1 - 3\mathbf{v}_2) = \dots \mathbf{w}_1 + \dots \mathbf{w}_2 + \dots \mathbf{w}_3$$

**Problem 6** Let

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{w}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let  $V$  and  $W$  be subspaces of  $\mathbb{R}^3$  with ordered bases

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\} \quad \text{and} \quad \mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2\}$$

respectively.

Let  $T : V \rightarrow W$  be a linear transformation such that

$$T(\mathbf{v}_1) = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

1. Show that  $\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$  lie in  $W$  by expressing them as linear combinations of  $\mathbf{w}_1$  and  $\mathbf{w}_2$ .

$$\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \dots \mathbf{w}_1 + \dots \mathbf{w}_2$$

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$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \dots \mathbf{w}_1 + \dots \mathbf{w}_2$$

2. Find the matrix of  $T$  with respect to  $\mathcal{B}$  and  $\mathcal{C}$ .

$$\begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

3. Find the matrix of  $T^{-1}$  with respect to  $\mathcal{C}$  and  $\mathcal{B}$ .

$$\begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

4. Find  $T^{-1} \left( \begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix} \right)$ .

$$T^{-1} \left( \begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$