
1 Addition and Scalar Multiplication of Matrices

Introduction to Matrices

A *matrix* is a rectangular array of numbers. The plural form of *matrix* is *matrices*. You have encountered matrices before in the context of augmented matrices and coefficient matrices associate with linear systems.

Consider the matrix

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix}$$

The *dimension of a matrix* is defined as $m \times n$ where m is the number of rows and n is the number of columns. The above matrix is a 3×4 matrix because there are three rows and four columns.

A *column vector* in $\mathbb{R}^n$ is an $n \times 1$ matrix. A *row vector* in $\mathbb{R}^n$ is a $1 \times n$ matrix.

The individual *entries* in the matrix are identified according to their position. The (i, j) -entry of a matrix is the entry in the i^{th} row and j^{th} column. For example, in matrix M above, 8 is called the $(2, 3)$ -entry because it is in the second row and the third column.

We denote the entry in the i^{th} row and the j^{th} column of matrix A by a_{ij} , and write A in terms of its entries as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Occasionally it will be convenient to talk about columns and rows of a matrix A as vectors. We will use the following notation:

$$A = \begin{bmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \\ | & | & & | \end{bmatrix} \quad \text{or} \quad A = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \dots \quad \mathbf{c}_n]$$

$$A = \begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_m & - \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$$

A matrix is called a *square matrix* if it has the same number of rows and columns. If $B = [b_{ij}]$ is an $n \times n$ square

matrix, the entries of the form b_{ii} are said to lie on the *main diagonal*. For example, if

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 4 & 9 \end{bmatrix}$$

then the main diagonal consists of entries $b_{11} = 1$, $b_{22} = 5$ and $b_{33} = 9$.

There are various operations which are done on matrices of appropriate sizes. Matrices can be added to and subtracted from other matrices, multiplied by a scalar, and multiplied by other matrices. We will never divide a matrix by another matrix, but we will see later how multiplication by a matrix inverse (if an inverse exists) plays a similar role to division.

In doing arithmetic with matrices, we often define the action by what happens in terms of the entries (or components) of the matrices. Before looking at these operations in depth, consider a few general definitions.

Definition 1 (The Zero Matrix). The $m \times n$ zero matrix is the $m \times n$ matrix having every entry equal to zero. The zero matrix is denoted by O .

Definition 2 (Equality of Matrices). Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then $A = B$ means that $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Addition of Matrices

Given two matrices of the same dimensions, we can add them together by adding their corresponding entries.

Definition 3 (Addition of Matrices). Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then the *sum of matrices* A and B , denoted by $A + B$, is an $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}]$$

Example 1. Find the sum of A and B , if possible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix}$$

Explanation. Notice that both A and B are of size 2×3 . Since A and B are of the same size, addition is possible.

$$\begin{aligned} A + B &= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+5 & 2+2 & 3+3 \\ 1+(-6) & 0+2 & 4+1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 4 & 6 \\ -5 & 2 & 5 \end{bmatrix} \end{aligned}$$

Going forward, whenever we write $A + B$ it will be assumed that the two matrices are of equal size and addition is possible.

Theorem 1 (Properties of Matrix Addition). Let A, B and C be matrices. Then, the following properties hold.

1. Commutative Law of Addition

$$A + B = B + A$$

2. Associative Law of Addition

$$(A + B) + C = A + (B + C)$$

3. Additive Identity

There exists a zero matrix such that

$$A + O = A$$

4. Additive Inverse

There exists a matrix, $-A$, such that

$$A + (-A) = O$$

We will prove Properties ?? and ??. The remaining properties are left as exercises.

Proof of Property ??: The (i, j) -entry of $A + B$ is given by

$$a_{ij} + b_{ij}$$

The (i, j) -entry of $B + A$ is given by

$$b_{ij} + a_{ij}$$

Since $a_{ij} + b_{ij} = b_{ij} + a_{ij}$, for all i, j , we conclude that $A + B = B + A$. ■

Proof of Property ??: Let $-A$ be defined by

$$-A = \begin{bmatrix} -a_{ij} \end{bmatrix}$$

Then $A + (-A) = O$. ■

You will recognize the zero matrix of Theorem ???? as the zero matrix of Definition ??.

Scalar Multiplication of Matrices

When a matrix is multiplied by a scalar, the new matrix is obtained by multiplying every entry of the original matrix by the given scalar.

Definition 4 (Scalar Multiplication of Matrices). If $A = [a_{ij}]$ and k is a scalar, then $kA = [ka_{ij}]$.

Example 2. Find $7A$ if

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}$$

Explanation. By Definition ??, we multiply each entry of A by 7. Therefore,

$$7A = 7 \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 7(2) & 7(0) \\ 7(1) & 7(-4) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 7 & -28 \end{bmatrix}$$

Theorem 2 (Properties of Scalar Multiplication). Let A, B be matrices, and k, p be scalars. Then, the following properties of scalar multiplication hold.

1. Distributive Law over Matrix Addition

$$k(A + B) = kA + kB$$

2. Distributive Law over Scalar Addition

$$(k + p)A = kA + pA$$

3. Associative Law for Scalar Multiplication

$$k(pA) = (kp)A$$

4. Multiplication by 1

$$1A = A$$

The proof of this theorem is similar to the proof of Theorem ?? and is left as an exercise.

Practice Problems

Problem 1 If

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$$

then

$$2(A + B) = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

$$3A - 2B = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Text Source

The text in this section is an adaptation of Section 2.1 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)
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