
1 Matrix Multiplication

We will introduce matrix multiplication by first considering the special case of a matrix-vector product. In other words, the first matrix is $m \times n$ and the second matrix is $n \times 1$ for some positive integers m, n .

Matrix-Vector Multiplication

Example 1. Let

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -2 & 1 \\ -2 & 4 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

One way to understand the matrix-vector product $A\mathbf{x}$ is by thinking of it as a linear combination of the columns of A , using the entries in $\mathbf{x}$ as our coefficients.

$$A\mathbf{x} = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -2 & 1 \\ -2 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 21 \\ -10 \\ -6 \end{bmatrix}$$

We can also compute the product one entry at a time. First, let's focus on the first row of A .

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -2 & 1 \\ -2 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} (2)(3) + (-1)(-1) + (3)(4) + (2)(1) \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 21 \\ \\ \\ \end{bmatrix}$$

Next, let's look at the second row of A .

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -2 & 1 \\ -2 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ (0)(3) + (3)(-1) + (-2)(4) + (1)(1) \\ \\ \end{bmatrix} = \begin{bmatrix} 21 \\ -10 \\ \\ \end{bmatrix}$$

Finally, let's do the third row of A .

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -2 & 1 \\ -2 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ -10 \\ (-2)(3) + (4)(-1) + (1)(4) + (0)(1) \end{bmatrix} = \begin{bmatrix} 21 \\ -10 \\ -6 \end{bmatrix}$$

Definition 1. Let A be an $m \times n$ matrix, and let $\mathbf{x}$ be an $n \times 1$ vector. The product $A\mathbf{x}$ is the $m \times 1$ vector given by:

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

or, equivalently,

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}$$

We can now make a couple of observations about the matrix-vector product. The first observation is part of the definition, but it is still worth pointing out.

Observation 1. In order for the product $A\mathbf{x}$ to exist, A and $\mathbf{x}$ must have compatible dimensions. In particular, vector $\mathbf{x}$ must have as many components as the number of columns of A . (Otherwise, we would not be able to have a well-defined linear combination of the columns.) So, if A is an $m \times n$ matrix, $\mathbf{x}$ must be an $n \times 1$ vector. If we write these dimensions next to each other, we will notice that the inner dimensions (n) must match, while the outer dimensions, m and 1 , give us the dimensions of the product.

$$\begin{array}{ccc} A & \mathbf{x} & = & A\mathbf{x} \\ (m \times n) & (n \times 1) & & (m \times 1) \end{array}$$

Observation 2. If you are familiar with the *dot product* (see [Dot Product and its Properties](#)), you may have noticed that each individual entry in the product matrix $A\mathbf{x}$ is the dot product of a row of A with $\mathbf{x}$. Thus, if the rows of A are vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n$ we can restate Definition ?? as follows:

$$A\mathbf{x} = \begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_n \cdot \mathbf{x} \end{bmatrix}$$

Let's find another matrix-vector product.

Example 2. Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -2 & 1 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Find $A\mathbf{x}$.

Explanation. Fill in the blanks below.

$$A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -2 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

Matrix-Matrix Multiplication

Matrix-matrix multiplication is simply an extension of the idea of matrix-vector multiplication. In order for the product definition to work, matrix dimensions must be compatible. Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix, then the product AB will be an $m \times p$ matrix.

$$\begin{array}{ccc} A & B & = & AB \\ (m \times n) & (n \times p) & & (m \times p) \end{array}$$

Just like with vector products, the inner dimensions must be the same, while the outer dimensions, m and p , give us the dimensions of the product.

Definition 2. Let A be an $m \times n$ matrix whose rows are vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$. Let B be an $n \times p$ matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$. Then the entries of the matrix product AB are given by the dot products

$$\begin{aligned} AB &= \begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_i & - \\ & \vdots & \\ - & \mathbf{r}_m & - \end{bmatrix} \begin{bmatrix} | & | & & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_j & \dots & \mathbf{b}_p \\ | & | & & | & | \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b}_1 & \mathbf{r}_1 \cdot \mathbf{b}_2 & \dots & \mathbf{r}_1 \cdot \mathbf{b}_j & \dots & \mathbf{r}_1 \cdot \mathbf{b}_p \\ \mathbf{r}_2 \cdot \mathbf{b}_1 & \mathbf{r}_2 \cdot \mathbf{b}_2 & \dots & \mathbf{r}_2 \cdot \mathbf{b}_j & \dots & \mathbf{r}_2 \cdot \mathbf{b}_p \\ \vdots & \vdots & & \vdots & & \vdots \\ \mathbf{r}_i \cdot \mathbf{b}_1 & \mathbf{r}_i \cdot \mathbf{b}_2 & \dots & \mathbf{r}_i \cdot \mathbf{b}_j & \dots & \mathbf{r}_i \cdot \mathbf{b}_p \\ \vdots & \vdots & & \vdots & & \vdots \\ \mathbf{r}_m \cdot \mathbf{b}_1 & \mathbf{r}_m \cdot \mathbf{b}_2 & \dots & \mathbf{r}_m \cdot \mathbf{b}_j & \dots & \mathbf{r}_m \cdot \mathbf{b}_p \end{bmatrix} \end{aligned}$$

So the (i, j) -entry of AB is the dot product of the i^{th} row of A and the j^{th} column of B .

In terms of components, if the i^{th} row of A is

$$\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}$$

and the j^{th} column of B is

$$\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

then the (i, j) -entry of AB is given by

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj} \quad (1)$$

Example 3. Let

$$A = \begin{bmatrix} 3 & 2 & -1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 4 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ -1 & 4 \\ 2 & 3 \\ 2 & 0 \end{bmatrix}$$

Find AB .

Explanation. Observe that A is a 3×4 matrix and B is a 4×2 matrix. So, we expect the product AB to be a 3×2 matrix.

$$\begin{aligned} AB &= \begin{bmatrix} 3 & 2 & -1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 4 \\ 2 & 3 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (3)(1) + (2)(-1) + (-1)(2) + (1)(2) & (3)(-2) + (2)(4) + (-1)(3) + (1)(0) \\ (0)(1) + (3)(-1) + (1)(2) + (1)(2) & (0)(-2) + (3)(4) + (1)(3) + (1)(0) \\ (1)(1) + (4)(-1) + (-1)(2) + (0)(2) & (1)(-2) + (4)(4) + (-1)(3) + (0)(0) \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 15 \\ -5 & 11 \end{bmatrix} \end{aligned}$$

Remark 1. It is possible to use linear combinations rather than dot products to compute a matrix-matrix product. If the columns of matrix B are given by $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$, then the matrix product consists of p columns, each of which is a matrix-vector product:

$$AB = \begin{bmatrix} | & | & \dots & | \\ A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \\ | & | & \dots & | \end{bmatrix}.$$

If we return to Example ?? above, we may use this approach to compute

$$A\mathbf{b}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$$

and

$$A\mathbf{b}_2 = -2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + 0 = \begin{bmatrix} -1 \\ 15 \\ 11 \end{bmatrix}.$$

Example 4. Let

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -2 & 3 & -2 & 2 \\ 1 & -3 & 1 & 1 & 0 \end{bmatrix}$$

Find AB .

Explanation. Fill in the blanks below.

$$AB = \begin{bmatrix} -2 & 1 \\ 3 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 3 & -2 & 2 \\ 1 & -3 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Example 5. Suppose that we know that $\mathbf{x} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ satisfies

$$\begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 19 \\ -13 \end{bmatrix}$$

Use this information to express $\begin{bmatrix} 19 \\ -13 \end{bmatrix}$ as a linear combination of the columns of $\begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}$.

Explanation.

$$-2 \begin{bmatrix} -2 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 19 \\ -13 \end{bmatrix}$$

Properties of Matrix Multiplication

Exploration 1 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Observe that both AB and BA are defined, and both products are 2×2 matrices. Let's compute the two products

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Clearly $AB \neq BA$. We say that A and B do not commute.

While it is possible to find specific matrices that commute, matrix multiplication is not commutative in general.

Warning 1. Matrix multiplication is not commutative.

One example of an $n \times n$ square matrix that commutes with all $n \times n$ matrices is the matrix I_n defined by

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

I_n has 1's along the main diagonal and 0's everywhere else. It is often useful to think of I_n as a matrix whose j^{th} column (and j^{th} row) is $\mathbf{e}_j$, the j^{th} standard unit vector of $\mathbb{R}^n$. When the dimensions of I_n are clear from the context, or irrelevant, we will omit the subscript n and simply refer to this matrix as I .

You can easily convince yourself that I commutes with all square matrices of appropriate dimensions. Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Verify that $AI = A$ and $IA = A$.

Because I acts like the multiplicative identity 1 in regular multiplication, I (or I_n) is called the *identity matrix*.

Next we list several important properties of matrix multiplication. These properties hold only when matrix sizes are such that the products are defined.

Theorem 1 (Properties of Matrix Multiplication). The following hold for matrices A, B , and C and for scalar c ,

1. Left Distributive Property

$$A(B + C) = AB + AC$$

2. Right Distributive Property

$$(B + C)A = BA + CA$$

3. Associativity

$$A(BC) = (AB)C$$

- 4.

$$c(AB) = (cA)B = A(cB)$$

5. Multiplicative Identity

$$AI = IA = A$$

Proof We prove ?? using the expression in (??) for the (i, j) -entry of a matrix product. (The proof of ?? is similar.) The (i, j) -entry of $A(B + C)$ is given by

$$\sum_k a_{ik}(b_{kj} + c_{kj}) = \sum_k a_{ik}b_{kj} + \sum_k a_{ik}c_{kj}$$

We recognize the right hand side as the (i, j) -entry of $AB + AC$. Thus $A(B + C) = AB + AC$.

For ??,

$$c \left(\sum_k a_{ik} b_{kj} \right) = \sum_k (ca_{ik}) b_{kj} = \sum_k a_{ik} (b_{kj}c).$$

For ??, the (i, j) -entry of the product AI is given by the dot product of the i^{th} row of A with the standard unit vector $\mathbf{e}_j$. Clearly, this dot product is a_{ij} . Because the (i, j) -entry of the product AI is equal to the (i, j) -entry A , we conclude that $AI = A$. The proof that $IA = A$ is similar.

Note that we skipped the proof of ??, which is quite cumbersome using sigma notation. We will easily tackle this proof later in the course when we cover [Composition of Linear Transformations](#). ■

Practice Problems

Problem 2 Explain why the following product is not defined.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Problem 3 Express the given product as a linear combination of the columns of the matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -5 \\ 7 \end{bmatrix} = \dots \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \dots \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \dots \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \dots \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Problem 4 Find the dimensions of the product.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Dimensions of product: $\dots \times \dots$

Problem 5 Find the dimensions of the product.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

Dimensions of product: $\dots \times \dots$

Problem 6 Find the product.

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ -2 & 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

Problem 7 Find the product.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 & 1 \\ 1 & 1 & -1 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots \end{bmatrix}$$

Problem 8 Find the product.

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 3 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} \dots & \dots \\ \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Text Source

The section on Properties of Matrix Multiplication is an adaptation of Section 2.1 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 67-68.