
1 Transpose of a Matrix

Another important operation on matrices is that of taking the *transpose*. For a matrix A , we denote the *transpose of A* by A^T . Before formally defining the transpose, we explore this operation on the following matrix.

$$\begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \end{bmatrix}$$

What happened? The first column became the first row and the second column became the second row. Thus the 3×2 matrix became a 2×3 matrix. The number 4 was in the first row and the second column and it ended up in the second row and first column.

The definition of the transpose is as follows.

Definition 1 (The Transpose of a Matrix). Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then the *transpose of A* , denoted by A^T , is the $n \times m$ matrix given by

$$A^T = [a_{ij}]^T = [a_{ji}]$$

The (i, j) -entry of A becomes the (j, i) -entry of A^T .

Example 1. Calculate A^T for the following matrix

$$A = \begin{bmatrix} 1 & 2 & -6 \\ 3 & 5 & 4 \end{bmatrix}$$

Explanation.

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -6 & 4 \end{bmatrix}$$

Note that A is a 2×3 matrix, while A^T is a 3×2 matrix. The columns of A are the rows of A^T , and the rows of A are the columns of A^T .

Theorem 1 (Properties of the Transpose of a Matrix). Let A be an $m \times n$ matrix, B an $n \times p$ matrix, and k a scalar. Then

1. $(A^T)^T = A$
2. $(AB)^T = B^T A^T$ (Shoes and Socks rule)
3. $(A + B)^T = A^T + B^T$
4. $(kA)^T = kA^T$

We will prove Property ???. The remaining properties are left as exercises.

Proof of Property ??: Note that A and B have compatible dimensions, so that AB is defined and has dimensions $m \times p$. Thus, $(AB)^T$ has dimensions $p \times m$. On the right side of the equality, A^T has dimensions $n \times m$, and B^T has dimensions $p \times n$. Therefore $B^T A^T$ is defined and has dimensions $p \times m$. Now we know that $(AB)^T$ and $B^T A^T$ have the same dimensions.

To show that $(AB)^T = B^T A^T$ we need to show that their corresponding entries are equal. Recall that the (i, j) -entry of AB is given by the dot product of the i^{th} row of A and the j^{th} column of B . The same dot product is also the (j, i) -entry of $(AB)^T$.

The (j, i) -entry of $B^T A^T$ is given by the dot product of the j^{th} row of B^T and the i^{th} column of A^T . But the j^{th} row of B^T has the same entries as the j^{th} column of B , and the i^{th} column of A^T has the same entries as the i^{th} row of A . Therefore the (j, i) -entry of $B^T A^T$ is also equal to the (i, j) -entry of AB .

Thus, the corresponding components of $(AB)^T$ are equal and we conclude that $(AB)^T = B^T A^T$. ■

Symmetric and Skew Symmetric Matrices

Definition 2 (Symmetric and Skew Symmetric Matrices). An $n \times n$ matrix A is said to be *symmetric* if $A = A^T$. It is said to be *skew symmetric* if $A = -A^T$.

We will explore these definitions in the following examples.

Example 2. Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Show that A is symmetric.

Explanation.

$$A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Hence, $A = A^T$, so A is symmetric.

Example 3. Let

$$A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$$

Show that A is skew symmetric.

Explanation.

$$A^T = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

Each entry of A^T is equal to -1 times the same entry of A . Hence, $A^T = -A$ and so by Definition ??, A is skew symmetric.

A special case of a symmetric matrix is a *diagonal matrix*. A diagonal matrix is a square matrix whose entries outside of the main diagonal are all zero. The identity matrix I is a diagonal matrix. Here is another example.

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Practice Problems

Problem 1 Prove Properties ??, ?? and ?? of Theorem ??.

Problem 2 Let A be an arbitrary matrix. What can you say about the dimensions of the product $A^T A$?

Problems ??-??

Classify each matrix as symmetric, skew symmetric, or neither.

Problem 3

$$\begin{bmatrix} 0 & -1 & -3 \\ 1 & -1 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

Multiple Choice:

1. Symmetric
2. Skew symmetric
3. Neither

Problem 4

$$\begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

Multiple Choice:

1. Symmetric
2. Skew symmetric
3. Neither

Problem 5

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

Multiple Choice:

1. *Symmetric*
2. *Skew symmetric*
3. *Neither*

Problem 6 *Give your own example of a 4×4 skew symmetric matrix.*

Problem 7 *Make a conjecture about the main diagonal entries of a skew symmetric matrix. Prove your conjecture.*

Text Source

The text in this section is an adaptation of Section 2.1 of Ken Kuttler's [*A First Course in Linear Algebra*](#). (CC-BY)
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