
1 Linear Systems as Matrix and Linear Combination Equations

A Linear System as a Matrix Equation

Exploration 1 Consider the linear system

$$\begin{array}{ccccccccc} 3x_1 & - & 2x_2 & + & 4x_3 & + & x_4 & = & 5 \\ -x_1 & & & & + & 5x_3 & - & 2x_4 & = & 1 \\ 2x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & -4 \end{array}$$

Let's construct the coefficient matrix A and multiply it by $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ on the right.

$$A\mathbf{x} = \begin{bmatrix} 3 & -2 & 4 & 1 \\ -1 & 0 & 5 & -2 \\ 2 & 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 + 4x_3 + x_4 \\ -x_1 + 5x_3 - 2x_4 \\ 2x_1 + x_2 - x_3 + 3x_4 \end{bmatrix}$$

Observe that each component of the product vector corresponds to one of the equations in the system. Let $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$. Then

$$A\mathbf{x} = \mathbf{b}$$
$$\begin{bmatrix} 3 & -2 & 4 & 1 \\ -1 & 0 & 5 & -2 \\ 2 & 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$$

is a matrix equation that corresponds to our system of equations.

In general, a system of linear equations

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

can be written as a matrix equation as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Solving this matrix equation (or showing that a solution does not exist) amounts to finding the reduced row-echelon form of the augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} & b_m \end{array} \right]$$

Example 1. Given a linear system

$$\begin{array}{rcrcrcrcrcl} x & + & 2y & = & 0 \\ -x & + & y & = & -3 \\ & & y & = & -1 \\ x & & & = & 2 \end{array}$$

1. Write the system as a matrix equation
2. Solve the system and the matrix equation

Explanation. ?? The matrix equation that corresponds to the system is

$$\begin{bmatrix} \dots & \dots \\ \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

?? The augmented matrix that corresponds to the original system and its reduced row-echelon form are

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 1 & -3 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

This shows that the ordered pair $(2, -1)$ is a solution to the system. We conclude that $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a solution to the matrix equation in ?? . A quick verification confirms this

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -1 \\ 2 \end{bmatrix}$$

Example 2. Let

$$A = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Solve $A\mathbf{x} = \mathbf{b}$.

Explanation. We write the equation $A\mathbf{x} = \mathbf{b}$ in augmented matrix form and apply elementary row operations to find its reduced row-echelon form.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 3 & -2 \end{array} \right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 1 & 4 & -4 \end{array} \right]$$

One way to obtain a solution is to convert this to a system of equations. It is not necessary to write the system down, but it helps to think about it as you write out your solution vector.

$$\begin{array}{ccccccccc} x_1 & & & - & x_3 & - & x_4 & = & 2 \\ & x_2 & + & x_3 & + & 4x_4 & = & -4 \end{array}$$

We see that x_1 and x_2 are leading variables because they correspond to leading 1s in the reduced row-echelon form, while x_3 and x_4 are free variables. We start by assigning parameters s and t to x_3 and x_4 , respectively, then solve for x_1 and x_2 .

$$\begin{aligned} x_1 &= 2 + s + t \\ x_2 &= -4 - s - 4t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

We can now write the solution vector as follows

$$\mathbf{x} = \begin{bmatrix} 2 + s + t \\ -4 - s - 4t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} t \quad (1)$$

The solution given in (1) is an example of a *general solution* because it accounts for all of the solutions to the system. Letting s and t take on specific values produces *particular solutions*. For example, $\begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ is a particular solution that corresponds to $s = 1, t = -1$.

Singular and Nonsingular Matrices

Our examples so far involved non-square matrices. Square matrices, however, play a very important role in linear algebra. This section will focus on square matrices.

Example 3. Let

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Solve $A\mathbf{x} = \mathbf{b}$.

Explanation. We apply elementary row operations to bring the augmented matrix to its reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 2 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

We can immediately see that the solution vector is

$$\mathbf{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Observe that the left-hand side of the augmented matrix in Example ?? is the identity matrix I . This means that $\text{rref}(A) = I$.

The elementary row operations that carried A to I were not dependent on the vector $\mathbf{b}$. In fact, the same row reduction process can be applied to the matrix equation $A\mathbf{x} = \mathbf{b}$ for *any* vector $\mathbf{b}$ to obtain a unique solution.

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & a \\ 0 & 1 & 2 & b \\ 1 & 2 & 2 & c \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a^* \\ 0 & 1 & 0 & b^* \\ 0 & 0 & 1 & c^* \end{array} \right]$$

$$\mathbf{x} = \begin{bmatrix} a^* \\ b^* \\ c^* \end{bmatrix}$$

Given a matrix A such that $\text{rref}(A) = I$, the system $A\mathbf{x} = \mathbf{b}$ will never be inconsistent because we will never have a row like this: $\left[\begin{array}{ccc|c} 0 & 0 & \dots & 0 & 1 \end{array} \right]$. Neither will we have infinitely many solutions because there will never be free variables. Matrices such as A deserve special attention.

Definition 1. A square matrix A is said to be *nonsingular* provided that $\text{rref}(A) = I$. Otherwise we say that A is *singular*.

Non-singular matrices have many useful properties.

Theorem 1. The following statements are equivalent for an $n \times n$ matrix A .

1. A is nonsingular
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$ in $\mathbb{R}^n$
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

We will prove equivalence of the three statements by showing that

$$?? \Rightarrow ?? \Rightarrow ?? \Rightarrow ??$$

Proof of $?? \Rightarrow ??$ Suppose $\text{rref}(A) = I$. Given any vector $\mathbf{b}$ in $\mathbb{R}^n$, the augmented matrix $[A|\mathbf{b}]$ can be carried to its reduced row-echelon form $[I|\mathbf{b}^*]$. Uniqueness of the reduced row-echelon form guarantees that $\mathbf{b}^*$ is the unique solution of $A\mathbf{x} = \mathbf{b}$. ■

Proof of $?? \Rightarrow ??$ Suppose $A\mathbf{x} = \mathbf{b}$ has a unique solution for all vectors $\mathbf{b}$. Then $A\mathbf{x} = \mathbf{0}$ has a unique solution. But $\mathbf{x} = \mathbf{0}$ is always a solution to $A\mathbf{x} = \mathbf{0}$. Therefore $\mathbf{x} = \mathbf{0}$ is the only solution. ■

Proof of $?? \Rightarrow ??$ Suppose $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This means that $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is the only solution of $A\mathbf{x} = \mathbf{0}$. But then, we know that the augmented matrix $[A|\mathbf{0}]$ can be reduced to $[I|\mathbf{0}]$. The same row operations will carry A to I . ■

Not all square matrices are nonsingular. For example,

$$\text{rref} \left(\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \neq I$$

By Theorem ??, a matrix equation $A\mathbf{x} = \mathbf{b}$ involving a singular matrix A cannot have a unique solution. The following example illustrates the two scenarios that arise when solving equations that involve singular matrices.

Example 4. Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$$

Solve the equation $A\mathbf{x} = \mathbf{b}$ or show that the equation is inconsistent.

1.

$$\mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

2.

$$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Explanation. ?? Row reduction gives us

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 4 \\ 1 & 1 & 1 & -1 \\ 3 & 0 & 2 & 3 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 2/3 & 1 \\ 0 & 1 & 1/3 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

There are infinitely many solutions.

$$\mathbf{x} = \begin{bmatrix} 1 - (2/3)t \\ -2 - (1/3)t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2/3 \\ -1/3 \\ 1 \end{bmatrix} t$$

?? When the vector $\mathbf{b}$ is changed, the row operations that take A to its reduced row-echelon form produce a 1 in the last row of the vector on the right, which shows that the system is inconsistent.

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 0 & 2 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 2/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

A Linear System as a Linear Combination Equation

Recall that the product of a matrix and a vector can be interpreted as a linear combination of the columns of the matrix. For example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Example 5. For each given matrix A and vector $\mathbf{b}$, determine whether $\mathbf{b}$ is a linear combination of the columns of A . If possible, express $\mathbf{b}$ as a linear combination of the columns of A .

1.

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 0 & 3 \\ -2 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 7 \\ -1 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 1 \\ 0 & 1 & -5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

Explanation. ?? We are looking for x_1, x_2, x_3 such that

$$x_1 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ -1 \end{bmatrix}$$

Solving this equation amounts to finding $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $A\mathbf{x} = \mathbf{b}$. The augmented matrix corresponding to this equation, together with its reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & -2 \\ 1 & 0 & 3 & 7 \\ -2 & 1 & 1 & -1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\text{So, } \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} -2 \\ 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

?? We begin by attempting to solve the matrix equation $\mathbf{Ax} = \mathbf{b}$. The augmented matrix corresponding to this equation, together with its reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 2 & -1 & 1 & -1 \\ 0 & 1 & -5 & 2 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

This shows that this matrix equation has no solutions. We conclude that $\mathbf{b}$ is not a linear combination of the columns of A .

Practice Problems

Problem 2 Given a system of linear equations, write (a) the corresponding matrix equation, and (b) the corresponding linear combination equation. DO NOT SOLVE.

$$\begin{array}{rrcr} 3x & - & y & - & 2z & = & 4 \\ -x & & & + & z & = & -1 \\ & & -y & + & 5z & = & 0 \end{array}$$

Problem 3 Use an augmented matrix and elementary row operations to find coefficients x_1 and x_2 that make the expression true, or demonstrate that such coefficients do not exist.

$$x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

If coefficients x_1 and x_2 do not exist, enter NA in each answer box.

$$X_1 = \dots, X_2 = \dots$$

Problem 4 Use an augmented matrix and elementary row operations to find coefficients x_1 and x_2 that make the expression true, or demonstrate that such coefficients do not exist.

$$x_1 \begin{bmatrix} 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -8 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

If coefficients x_1 and x_2 do not exist, enter NA in each answer box.

$$X_1 = \dots\dots, X_2 = \dots\dots$$

Problem 5 Determine whether vector **b** is in the span of the given set of vectors.

$$\mathbf{b} = \begin{bmatrix} 2 \\ 14 \\ 7 \end{bmatrix}$$
$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} \right\}$$

Multiple Choice:

1. **b** is in the span of the vectors.
2. **b** is NOT in the span of the vectors.

Problem 6 Determine whether vector **b** is in the span of the given set of vectors.

$$\mathbf{b} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$
$$\left\{ \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Multiple Choice:

1. **b** is in the span of the vectors.
2. **b** is NOT in the span of the vectors.

Problem 7 Determine whether vector **b** is in the span of the given set of vectors.

$$\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ -7 \\ -5 \end{bmatrix}$$
$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \\ 1 \end{bmatrix} \right\}$$

Multiple Choice:

1. **b** is in the span of the vectors.
2. **b** is NOT in the span of the vectors.