
1 The Inverse of a Matrix

Definition and Properties of Matrix Inverses

Consider the equation $2x = 6$. It takes little time to recognize that the solution to this equation is $x = 3$. In fact, the solution is so obvious that we do not think about the algebraic steps necessary to find it. Let's take a look at these steps in detail.

$$\begin{aligned}2x &= 6 \\ \frac{1}{2} \times (2x) &= \frac{1}{2} \times 6 \\ \left(\frac{1}{2} \times 2\right)x &= 3 \\ 1x &= 3 \\ x &= 3\end{aligned}$$

This process utilizes many properties of real-number multiplication. In particular, we make use of existence of *multiplicative inverses*. Every non-zero real number a has a multiplicative inverse $a^{-1} = \frac{1}{a}$ with the property that $\frac{1}{a} \times a = a \times \frac{1}{a} = 1$. We say that 1 is the *multiplicative identity* because $a \times 1 = 1 \times a = a$.

Given a matrix equation $A\mathbf{x} = \mathbf{b}$, we would like to follow a process similar to the one above to solve this matrix equation for $\mathbf{x}$.

Observe that the role of the multiplicative identity for $n \times n$ square matrices is filled by I_n because $AI_n = I_nA = A$. Given an $n \times n$ matrix A , a multiplicative inverse of A would have to be some $n \times n$ matrix B such that

$$BA = AB = I_n$$

Assuming that such an inverse B exists, this is what the process of solving the equation $A\mathbf{x} = \mathbf{b}$ would look like:

$$\begin{aligned}A\mathbf{x} &= \mathbf{b} \\ B(A\mathbf{x}) &= B\mathbf{b} \\ (BA)\mathbf{x} &= B\mathbf{b} \\ I\mathbf{x} &= B\mathbf{b} \\ \mathbf{x} &= B\mathbf{b}\end{aligned}$$

Definition 1. Let A be an $n \times n$ matrix. An $n \times n$ matrix B is called an *inverse* of A if

$$AB = BA = I$$

where I is an $n \times n$ identity matrix. If such an inverse matrix exists, we say that A is *invertible*. If an inverse does not exist, we say that A is not invertible.

It follows directly from the way the definition is stated that if B is an inverse of A , then A is an inverse of B . We say that A and B are inverses of each other.

The following theorem shows that matrix inverses are unique.

Theorem 1. Suppose A is an invertible matrix, and B is an inverse of A . Then B is unique.

Proof Because B is an inverse of A , we have:

$$AB = BA = I$$

Suppose there exists another $n \times n$ matrix C such that

$$AC = CA = I$$

Then

$$C(AB) = CI$$

$$(CA)B = C$$

$$IB = C$$

$$B = C$$

■

Now that we know that a matrix A cannot have more than one inverse, we can safely refer to the inverse of A as A^{-1} .

Example 1. Let

$$A = \begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2/3 & -1/3 \\ -5/3 & -1/3 \end{bmatrix}$$

Verify that A and B are inverses of each other.

Explanation. We will show that $AB = BA = I$.

$$AB = \begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} -2/3 & -1/3 \\ -5/3 & -1/3 \end{bmatrix} = \begin{bmatrix} -2/3 + 5/3 & -1/3 + 1/3 \\ 10/3 - 10/3 & 5/3 - 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2/3 & -1/3 \\ -5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -2/3 + 5/3 & 2/3 - 2/3 \\ -5/3 + 5/3 & 5/3 - 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2. Use what we found in Example ?? to solve the matrix equation:

$$\begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Explanation. We multiply both sides of the equation by the inverse of $\begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix}$.

$$\begin{bmatrix} -2/3 & -1/3 \\ -5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -2/3 & -1/3 \\ -5/3 & -1/3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 5/3 \\ 14/3 \end{bmatrix}$$

We now prove several useful properties of matrix inverses.

Theorem 2. The following properties are stated for square matrices of appropriate sizes.

1. $I^{-1} = I$
2. For an invertible matrix A , $(A^{-1})^{-1} = A$.
3. If A and B are invertible matrices, then (AB) is invertible and $(AB)^{-1} = B^{-1}A^{-1}$ (Shoes and socks rule).
4. If A is an invertible matrix and k is a non-zero number, then $(kA)^{-1} = \frac{1}{k}A^{-1}$.
5. If A is an invertible matrix, then A^T is also invertible, and $(A^T)^{-1} = (A^{-1})^T$.

We will prove ???. The remaining properties are left as exercises.

Proof of Property ???: We will check to see if $B^{-1}A^{-1}$ is the inverse of AB .

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Thus (AB) is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. ■

Computing the Inverse

We now turn to the question of how to find the inverse of a square matrix, or determine that the inverse does not exist.

Given a square matrix A , we are looking for a square matrix B such that

$$AB = I \quad \text{and} \quad BA = I$$

We will start by attempting to satisfy $AB = I$. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the columns of B , then

$$A \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ | & | & & | \end{bmatrix}$$

where each $\mathbf{e}_i$ is a standard unit vector of $\mathbb{R}^n$. This gives us a system of equations $A\mathbf{v}_i = \mathbf{e}_i$ for each $1 \leq i \leq n$. If each $A\mathbf{v}_i = \mathbf{e}_i$ has a unique solution, then finding these solutions will give us the columns of the desired matrix B .

First, suppose that $\text{rref}(A) = I$, then we can use elementary row operations to carry each $[A|\mathbf{e}_i]$ to its reduced row-echelon form.

$$[A|\mathbf{e}_i] \rightsquigarrow [I|\mathbf{v}_i]$$

Observe that the row operations that carry A to I will be the same for each $A\mathbf{v}_i = \mathbf{e}_i$. We can, therefore, combine the process of solving n systems of equations into a single process

$$[A|I] \rightsquigarrow [I|B]$$

Each $\mathbf{v}_i$ is a unique solution of $A\mathbf{v}_i = \mathbf{e}_i$, and we conclude that

$$B = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

is a solution to $AB = I$.

By Problem ??, we can reverse the elementary row operations to obtain

$$[I|B] \rightsquigarrow [A|I]$$

But the same row operations would also give us

$$[B|I] \rightsquigarrow [I|A]$$

We conclude that $BA = I$, and $B = A^{-1}$.

Next, suppose that $\text{rref}(A) \neq I$. Then $\text{rref}(A)$ must contain a row of zeros. Because one of the rows of A was completely wiped out by elementary row operations, one of the rows of A must be a linear combination of the other rows. Suppose row p is a linear combination of the other rows. Then row p can be carried to a row of zeros. But then the system $A\mathbf{v}_p = \mathbf{e}_p$ is inconsistent. This is because $\mathbf{e}_p$ has a 1 as the p^{th} entry and zeros everywhere else. The 1 in the p^{th} spot will not be affected by elementary row operations, and the p^{th} row will eventually look like this

$$[0 \dots 0 | 1]$$

This shows that a matrix B such that $AB = I$ does not exist, and A does not have an inverse.

We have just proved the following theorem.

Theorem 3 (Row-reduction Method for Computing the Inverse of a Matrix). Let A be a square matrix. If it is possible to use elementary row operations to carry the augmented matrix $[A|I]$ to $[I|B]$, then $B = A^{-1}$. If such a reduction is not possible, then A does not have an inverse.

Corollary 1. A square matrix A has an inverse if and only if $\text{rref}(A) = I$.

Example 3. Find A^{-1} or demonstrate that A^{-1} does not exist.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Explanation. We start with the augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 1 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & 3 & -1 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1}} \begin{bmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 2 & -1 & | & -1 & 1 & 0 \\ 0 & 4 & -3 & | & -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - 2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array} \right] \xrightarrow{(-1)R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array} \right] \xrightarrow{\begin{array}{l} R_1 - 2R_3 \\ R_2 + R_3 \end{array}}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 3 & -4 & 2 \\ 0 & 2 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array} \right] \xrightarrow{\frac{1}{2}R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array} \right] \xrightarrow{R_1 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5/2 & 3/2 \\ 0 & 1 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array} \right]$$

We conclude that

$$A^{-1} = \begin{bmatrix} 2 & -5/2 & 3/2 \\ -1 & 3/2 & -1/2 \\ -1 & 2 & -1 \end{bmatrix}$$

Example 4. Find A^{-1} or demonstrate that A is not invertible.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 1 \\ 4 & 4 & -3 \end{bmatrix}$$

Explanation. We start with the augmented matrix

$$\left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 4 & 4 & -3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} \frac{1}{2}R_1 \\ \frac{1}{2}R_2 \end{array}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 4 & 4 & -3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 - 4R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & -2 & -1 & -2 & 0 & 1 \end{array} \right] \xrightarrow{R_3+2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array} \right]$$

At this point we see that the left-hand side cannot be turned into I through elementary row operations. We conclude that A^{-1} does not exist.

Remark 1. Recall that a square matrix whose reduced row-echelon form is the identity matrix is called *nonsingular*. (Definition ??) According to Corollary ??, a matrix is invertible if and only if it is nonsingular. For this reason many linear algebra texts use the terms *invertible* and *nonsingular* as synonyms.

Inverse of a 2×2 Matrix

We will conclude this section by discussing the inverse of a nonsingular 2×2 matrix. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a nonsingular matrix. We can find A^{-1} by using the row reduction method described above, that is, by computing the reduced row-echelon form of $[A|I]$. Row reduction yields the following:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & d/(ad-bc) & -b/(ad-bc) \\ 0 & 1 & -c/(ad-bc) & a/(ad-bc) \end{array} \right]$$

Note that the denominator of each term in the inverse matrix is the same. Factoring it out, gives us the following formula for A^{-1} .

Formula 1.

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Clearly, the expression for A^{-1} is defined, if and only if $ad-bc \neq 0$. So, what happens when $ad-bc = 0$? In Practice Problem ?? you will be asked to fill in the steps of the row reduction procedure that produces this formula, and show that if $ad-bc = 0$ then A does not have an inverse.

Practice Problems

Problem 1 Verify that the matrix $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ is its own inverse.

Problem 2 Use the row-reduction method for computing matrix inverses to explain why the given matrix does not have an inverse.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

Problem 3 Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1/12 & 1/6 & -1/4 \\ -1/4 & 1/2 & 1/4 \\ 5/12 & 1/6 & 1/4 \end{bmatrix}$$

Are A and B inverses of each other?

Multiple Choice:

1. Yes, A and B are inverses.
2. No, A and B are not inverses.

Problem 4 Use the following invertible matrices to illustrate Properties ??, ??, ?? and ?? of Theorem ??.

$$A = \begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}$$

Problem 5 In Example ?? we found the inverse of

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

to be

$$A^{-1} = \begin{bmatrix} 2 & -5/2 & 3/2 \\ -1 & 3/2 & -1/2 \\ -1 & 2 & -1 \end{bmatrix}$$

Use A^{-1} to solve the equation

$$A\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

Answer:

$$\mathbf{x} = \begin{bmatrix} \dots \dots \\ \dots \dots \\ \dots \dots \end{bmatrix}$$

Problem 6 Find the inverse of each matrix by using the row-reduction procedure.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 0 \\ 1 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Problem 7 Use Formula ?? to find the inverse of

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}$$

Use A^{-1} to solve the equation

$$A\mathbf{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Answer:

$$\mathbf{x} = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

Problem 8 Find the value of x for which the matrix is not invertible.

$$\begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix}$$

Answer: $x = \dots$

Problem 9 Suppose $AB = AC$ and A is an invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

Problem 10 Suppose $AB = AC$ and A is a non-invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

Problem 11 Suppose A and B are invertible $n \times n$ matrices. Does it follow that $(A + B)$ is invertible?