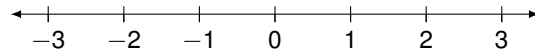

1 A Brief Introduction to $\mathbb{R}^n$

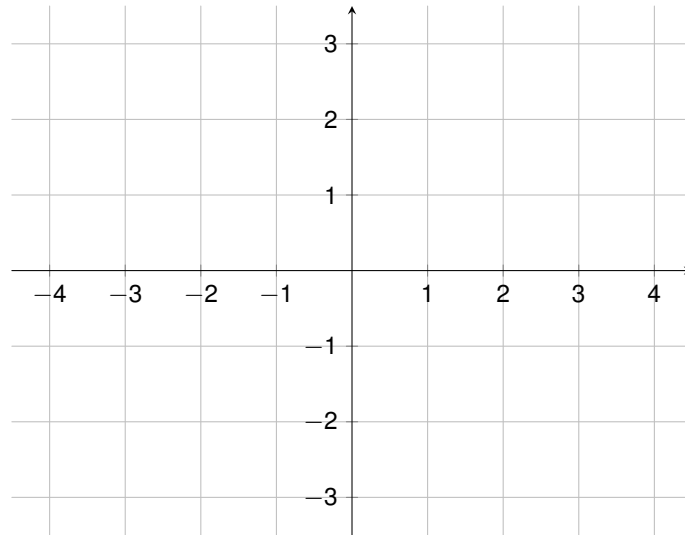
The set of all real numbers is denoted by $\mathbb{R}$. It is convenient to associate real numbers with points on a line, called the *real number line*.



The set of all ordered pairs (x, y) , where x and y are real numbers, is called $\mathbb{R}^2$. Using set notation we write:

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

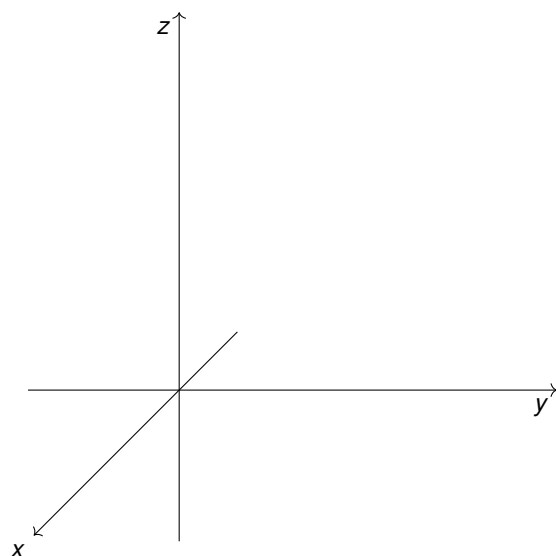
Geometrically speaking, $\mathbb{R}^2$ can be associated with a coordinate plane in which we refer to each point by its x and y coordinates.



The set of all ordered triples (x, y, z) , where x , y and z are real numbers, is called $\mathbb{R}^3$.

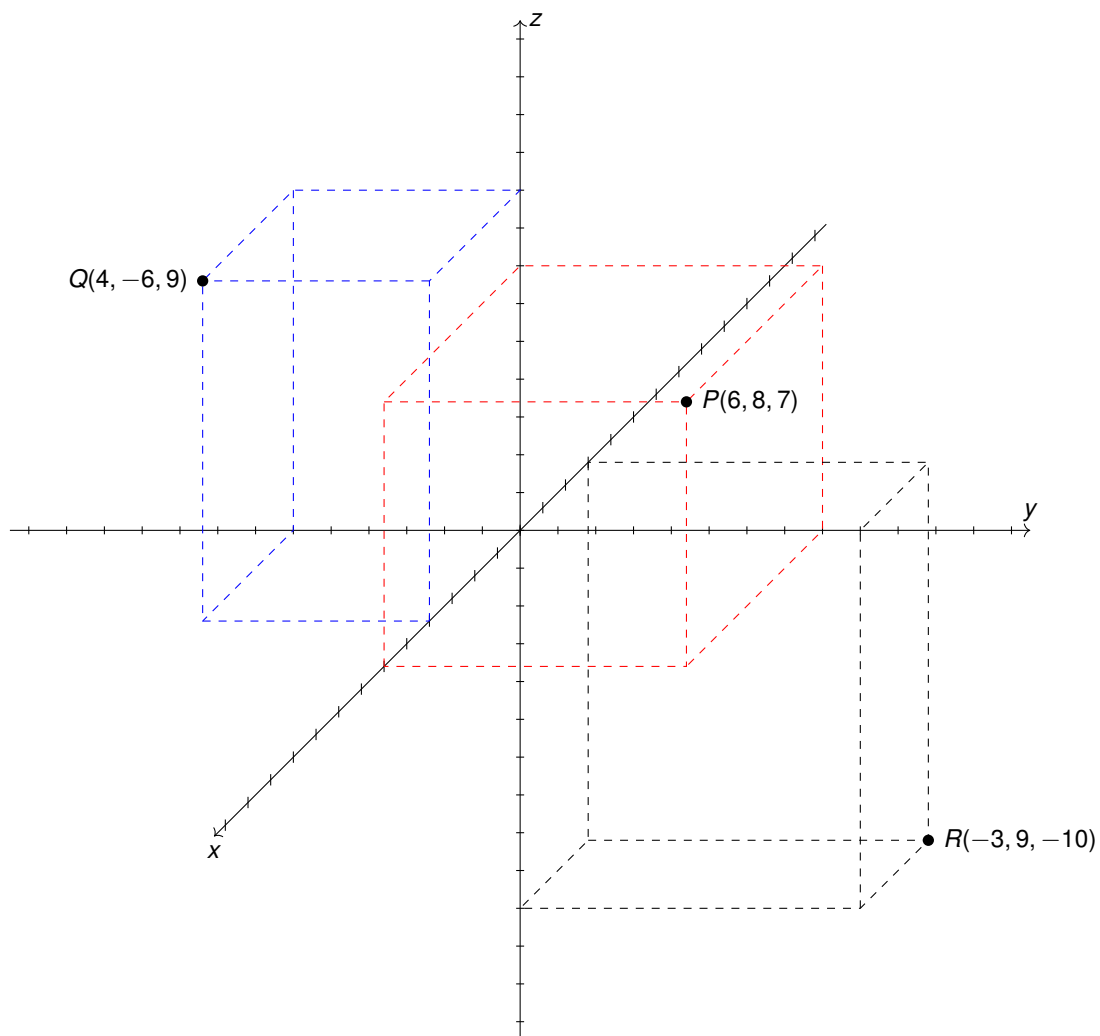
$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

Geometrically, an ordered triple of $\mathbb{R}^3$ is associated with a point of a three-dimensional space whose position is given by x , y and z coordinates.

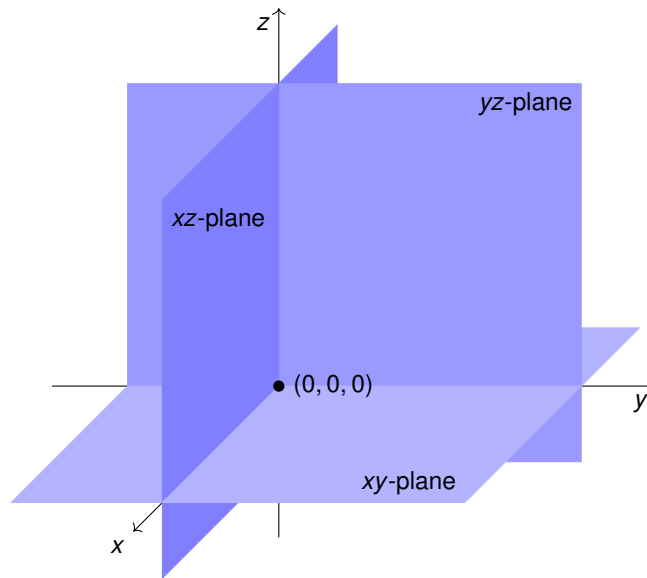


Example 1. The following points are shown plotted in $\mathbb{R}^3$.

1. $P(6, 8, 7)$
2. $Q(4, -6, 9)$
3. $R(-3, 9, -10)$



Each pair of axes in $\mathbb{R}^3$ determines a plane. The resulting three planes are called *coordinate planes*. Each coordinate plane is named after the axes that determine it. Thus, we have the *xy-plane*, *xz-plane*, and *yz-plane*. Coordinate planes intersect at the point $(0, 0, 0)$, called the *origin*, and subdivide $\mathbb{R}^3$ into eight regions, called *octants*.



The set of all ordered n -tuples $(x_1, x_2, \dots, x_n)$, where x_i is a real number for $1 \leq i \leq n$, is called $\mathbb{R}^n$.

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}$$

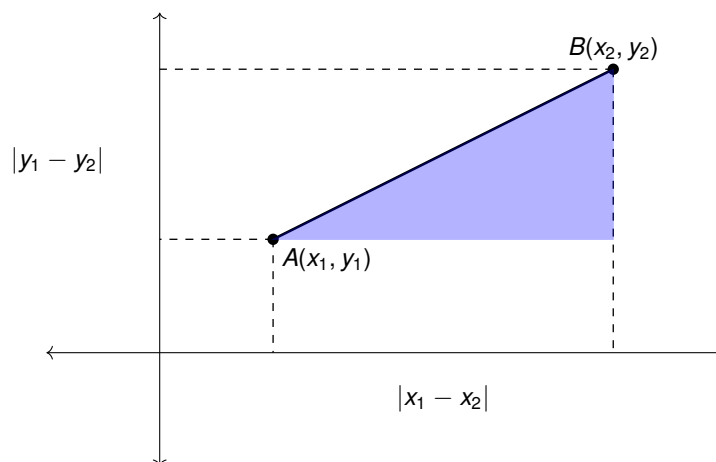
The point $(0, 0, \dots, 0)$ in $\mathbb{R}^n$ is called the *origin*.

$\mathbb{R}^n$ cannot be visualized for $n > 3$, but many familiar ideas, such as the distance formula, can be generalized to $\mathbb{R}^n$.

Distance in $\mathbb{R}^n$

In this section we will establish a formula for the distance between two points in $\mathbb{R}^n$. We begin by observing that the distance between two numbers (points) x_1 and x_2 on the number line is given by $|x_1 - x_2|$. (Why do we use the absolute value brackets?).

We can use the Pythagorean Theorem to establish the distance formula for points of $\mathbb{R}^2$.



Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be points in $\mathbb{R}^2$. By the Pythagorean Theorem we have

$$AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Why were we able to drop the absolute value brackets?

The distance formula for points in $\mathbb{R}^3$ can also be derived using the Pythagorean Theorem. Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be points of $\mathbb{R}^3$. Use the following GeoGebra interactive to walk through the steps of the derivation of the distance formula. RIGHT-CLICK and DRAG to rotate the image.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

Let points $A'(x_1, y_1, 0)$ and $B'(x_2, y_2, 0)$ be projections of A and B onto the xy -plane. By the distance formula in $\mathbb{R}^2$, the distance between A' and B' is

$$A'B' = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Let $C = (x_2, y_2, z_1)$. Observe that $\triangle ABC$ is a right triangle with $AC = A'B'$, and $BC = |z_1 - z_2|$. By the Pythagorean theorem we have

$$AB^2 = AC^2 + BC^2 = \left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \right)^2 + (z_1 - z_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

This gives us the following formula.

Formula 1. Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be points in $\mathbb{R}^3$. The distance between A and B is given by

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

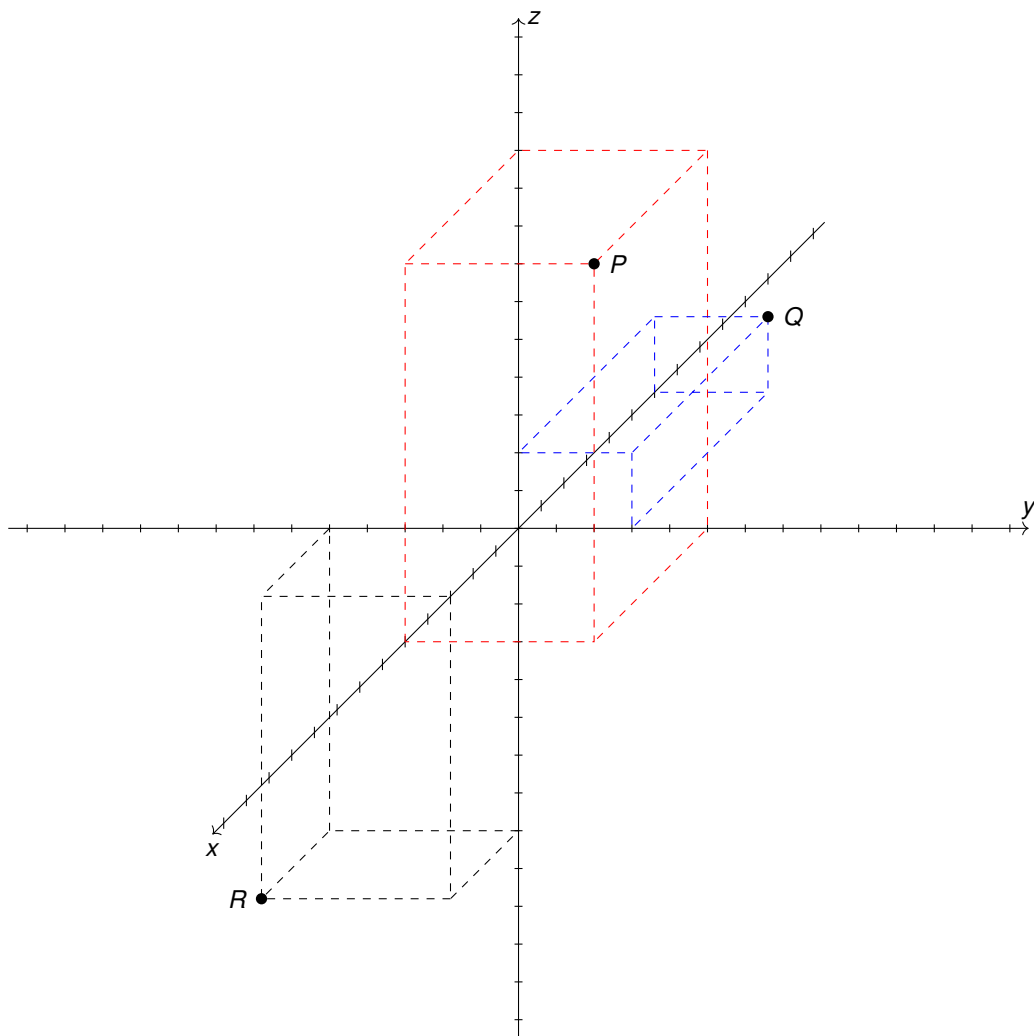
Observe the similarity of pattern in the distance formulas for $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$. We will take advantage of this pattern to define the distance between two points of $\mathbb{R}^n$.

Formula 2. Let $A(a_1, a_2, \dots, a_n)$ and $B(b_1, b_2, \dots, b_n)$ be points in $\mathbb{R}^n$. The distance between A and B is given by

$$AB = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \dots + (a_n - b_n)^2}$$

Practice Problems

Problem 1 Find the coordinates of each point.



Answer:

$P(\dots, \dots, \dots)$

$Q(\dots, \dots, \dots)$

$R(\dots, \dots, \dots)$

Problem 2 Find the coordinates of each point. RIGHT-CLICK and DRAG to rotate the image.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

Answer:

$A(\dots, \dots, \dots)$

$$B(\dots, \dots, \dots)$$

$$C(\dots, \dots, \dots)$$

$$D(\dots, \dots, \dots)$$

Problem 3 Find the distance between $A(-2, -1, 4)$ and $B(1, -5, -8)$.

$$AB = \dots$$