
1 Parametric Equations of Lines

At this point you should be very familiar with graphing linear equations of the form $y = mx + b$, where m is the slope of the line and b is the y -intercept. Unfortunately, there is no equivalent way of representing lines in $\mathbb{R}^3$, or better yet, $\mathbb{R}^n$. In this section we will develop an alternative way of representing lines.

Parametric Lines in $\mathbb{R}^2$

Exploration 1 *Imagine a ladybug crawling around a plane. At every instant in time, the ladybug's position in the plane can be described by an ordered pair (x, y) . Coordinates x and y are functions of time. Suppose the position of the ladybug at time t is given by:*

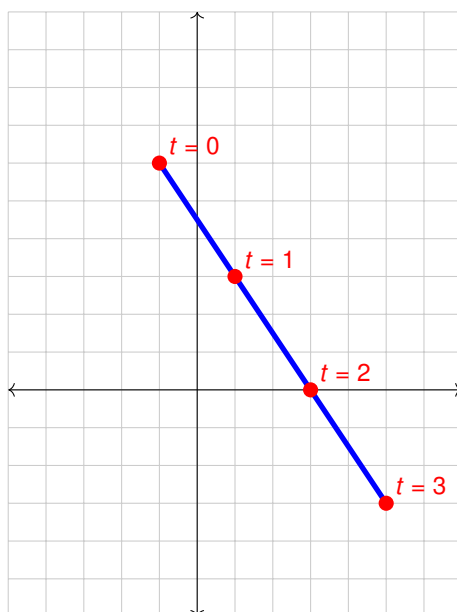
$$x = 2t - 1$$

$$y = -3t + 6$$

Our goal is to sketch the path of the ladybug during the time interval $0 \leq t \leq 3$. To do this we will make a table of values, just like you did when you first started sketching graphs of functions.

t	x	y
0	-1	6
1	1	3
2	3	0
3	5	-3

The point (x, y) , corresponding to each value of t is the location of the ladybug in the coordinate plane at time t .



It appears that the points corresponding to $t = 0, 1, 2, 3$ lie on a line with slope $-\frac{3}{2}$. In the next problem we will find the equation of the line that contains the path of the ladybug.

Exploration 2 In Exploration ?? we considered equations

$$x = 2t - 1$$

$$y = -3t + 6$$

that described the path of a ladybug in the plane. We will now consider these equations in a broader context as simply describing a curve in the plane. We conjectured that the curve described by these equations is a line with slope $-\frac{3}{2}$. We will now find the equation of this line.

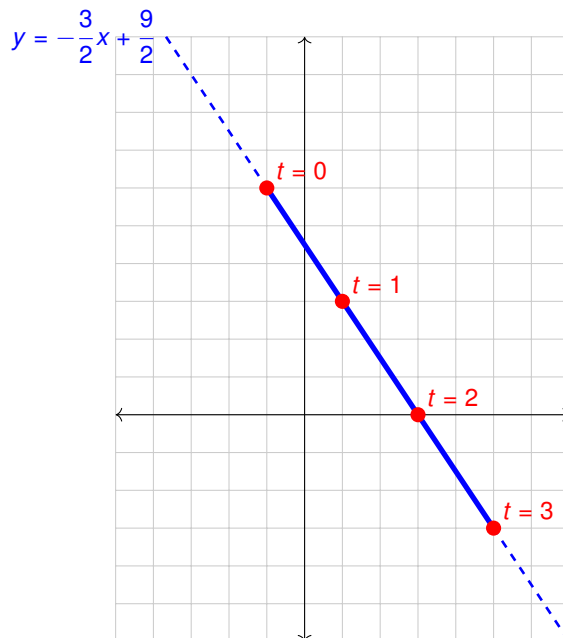
One approach to finding the equation is to solve one of the given equations for t , then substitute into the other equation. Solving $x = 2t - 1$ for t gives us

$$t = \frac{1}{2}x + \frac{1}{2}$$

Substituting this expression into $y = -3t + 6$ we get

$$y = -3\left(\frac{1}{2}x + \frac{1}{2}\right) + 6 = -\frac{3}{2}x + \frac{9}{2}$$

Plotting the line $y = -\frac{3}{2}x + \frac{9}{2}$ together with the path we plotted earlier, we see that our original path lies on the line.



If we do not restrict t to values between 0 and 3, we can get every point on the line $y = -\frac{3}{2}x + \frac{9}{2}$.

Equations such as

$$x = 2t - 1$$

$$y = -3t + 6$$

are called *parametric equations*, and t is called a *parameter*.

When given an equation of the form $y = mx + b$, we recognize it as an equation whose graph is a line and we don't need to make a table of values to sketch the graph of the equation. We should be able to do the same for parametric equations of lines. In the next Exploration we will examine our equations carefully to see if we can discern any patterns that would help us plot the line without making a table of values.

Exploration 3 *Let's return to parametric equations from the previous problem.*

$$x = 2t - 1$$

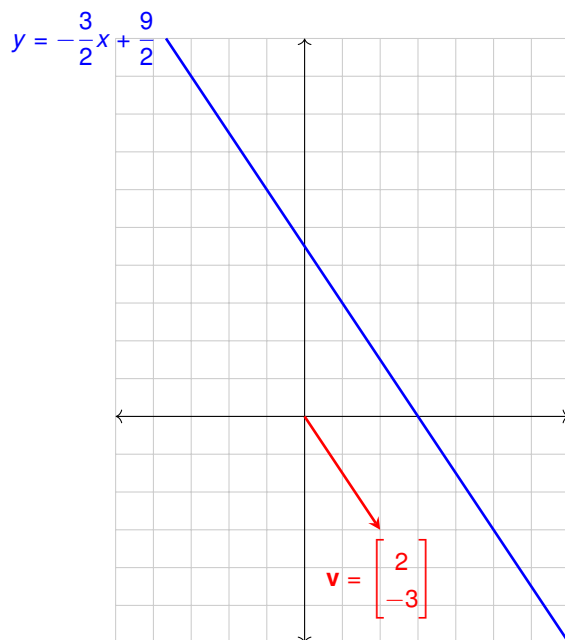
$$y = -3t + 6$$

Consider the table of values we constructed in Exploration ??.

t	x	y
0	-1	6
1	1	3
2	3	0
3	5	-3

Observe that every time t increases by 1, x increases by 2 and y decreases by 3. An increase by 2 in the x -coordinate combined with a decrease of 3 in the y -coordinate corresponds to the coefficients 2 and -3 of t in the parametric equations. Our findings are in agreement with the fact that the slope of the line is $-\frac{2}{3}$.

We will capture the “rise” and “run” aspect of the line by using vector $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Sketching $\mathbf{v}$ together with the line, we find that the vector is parallel to the line.



Vector $\mathbf{v}$ is called a direction vector. Observe that the components of $\mathbf{v}$ are the coefficients of t in the parametric equations.

$$x = 2t - 1$$

$$y = -3t + 6$$

Next, let's turn our attention to constants -1 and 6 . They are the values of x and y when $t = 0$, and correspond to the point $(-1, 6)$.

Recall that in Exploration ?? we said that the given parametric equations describe the position of a ladybug crawling in the coordinate plane. If we are concerned with the position of the ladybug at time t , the point $(-1, 6)$ is very important - this is where the bug is located when $t = 0$. But, if we are using parametric equations simply to describe the line $y = -\frac{3}{2}x + \frac{9}{2}$, without regard for when the bug is located at each point, then the point $(-1, 6)$ is not any more special than any other point on the line. In fact, we can use any other point on this line to find another set of parametric equations that describe the same line. In Practice Problem ?? you will be asked to show that equations

$$x = 2t + 1$$

$$y = -3t + 3$$

describe the same line. Thus, parametric representations are not unique.

We can generalize our observations in Exploration ?? as follows.

Formula 1. Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a direction vector for line l , and let (x_0, y_0) be an arbitrary point on l . Then the following parametric equations describe l :

$$x = at + x_0$$

$$y = bt + y_0$$

Example 1. Find parametric equations for line l if l contains the point $(2, -5)$ and is parallel to line k with parametric equations

$$x = 3t + 4$$

$$y = -8t + 1$$

Explanation. To find a set of parametric equations for l , we need a point on l and a direction vector. Point $(2, -5)$ is on l . Line l is parallel to k , so we can use $\begin{bmatrix} 3 \\ -8 \end{bmatrix}$, a direction vector for k , as a direction vector for l . This gives us the following parametric equations for l :

$$x = 3t + 2$$

$$y = -8t - 5$$

Parametric Lines in $\mathbb{R}^3$

Exploration 4 Consider the following set of parametric equations

$$x = 4t + 1$$

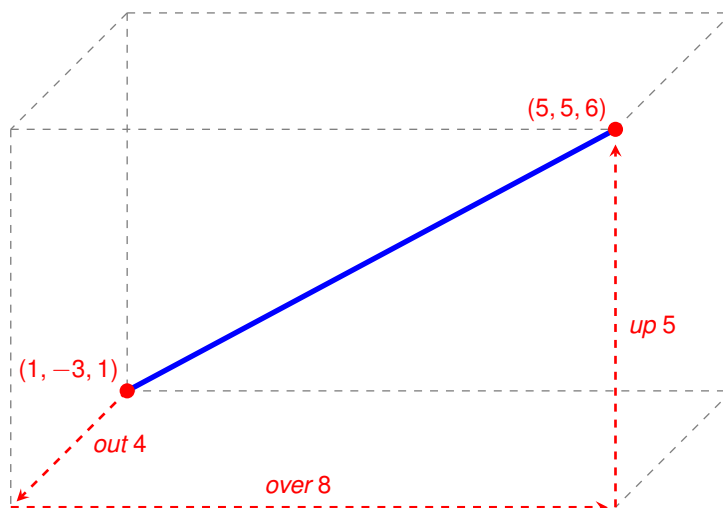
$$y = 8t - 3$$

$$z = 5t + 1$$

We will begin by making a table of values.

t	x	y	z
-1	-3	-11	-4
0	1	-3	1
1	5	5	6
2	9	13	11

Observe that every time t increases by 1, x increases by 4, y increases by 8, and z increases by 5. The pattern “out 4, over 8, up 5” is illustrated in the diagram.



We will use a direction vector $\mathbf{v} = \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}$ to represent the direction of the line. Note that the components of $\mathbf{v}$ are the same as the coefficients of t in the parametric equations.

$$\begin{aligned}x &= 4t + 1 \\y &= 8t - 3 \\z &= 5t + 1\end{aligned}$$

Also, observe that the constants $1, -3, 1$ correspond to the coordinates of the point on the line for which $t = 0$.

$$\begin{aligned}x &= 4t + 1 \\y &= 8t + (-3) \\z &= 5t + 1\end{aligned}$$

As before, the point $(1, -3, 1)$ may be replaced with any other point on the line to produce a different set of parametric equations describing the same line.

We can generalize our observations in Exploration ?? as follows.

Formula 2. Let $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a direction vector for line l , and let (x_0, y_0, z_0) be an arbitrary point on l . Then the following parametric equations describe l :

$$\begin{aligned}x &= at + x_0 \\y &= bt + y_0 \\z &= ct + z_0\end{aligned}$$

Example 2. Find a set of parametric equations for a line in $\mathbb{R}^3$ that passes through $(1, -2, 4)$ and $(-3, 1, 1)$.

Explanation. We need a point and a direction vector. To find the direction vector we will use the “head-tail” formula (Formula ??) to find the vector whose tail is at $(1, -2, 4)$ and whose head is at $(-3, 1, 1)$. The direction vector is

$$\begin{bmatrix} -3 - 1 \\ 1 - (-2) \\ 1 - 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ -3 \end{bmatrix}$$

For our point, we can pick either of the two given points. The equations will differ depending on the point we pick, but they will describe the same line. Remember, parametric representations are not unique! If we choose $(1, -2, 4)$ for our point we get the following set of parametric equations:

$$\begin{aligned}x &= -4t + 1 \\y &= 3t - 2 \\z &= -3t + 4\end{aligned}$$

Parametric Lines in $\mathbb{R}^n$

After working with parametric equations of lines in $\mathbb{R}^2$ and $\mathbb{R}^3$, it should be easy to surmise what parametric equations of lines look like for lines in $\mathbb{R}^n$.

Formula 3. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be a direction vector for line l in $\mathbb{R}^n$, and let $(a_1, a_2, \dots, a_n)$ be an arbitrary point on l . Then

the following parametric equations describe l :

$$x_1 = v_1 t + a_1$$

$$x_2 = v_2 t + a_2$$

$$\vdots$$

$$x_n = v_n t + a_n$$

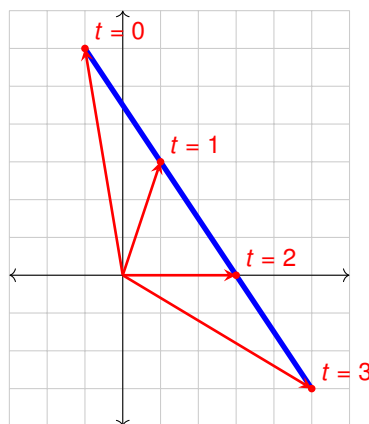
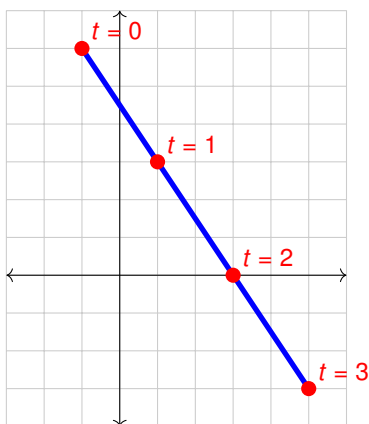
From Parametric Equations to Vector Equations

Consider again the parametric equations from Exploration ??.

$$x = 2t - 1$$

$$y = -3t + 6$$

These equations describe the line by providing coordinates of individual points on the line, as shown in the left figure below.



If we consider the points on the line to be heads of vectors, then we can think of the line as being “traced” by a vector as t changes (See above, right). We can capture this algebraically by using a *vector equation* for the line:

$$\mathbf{x}(t) = \begin{bmatrix} 2t - 1 \\ -3t + 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} t + \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

Note that the direction vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and the point $(-1, 6)$, now vector $\begin{bmatrix} -1 \\ 6 \end{bmatrix}$, are easy to identify in the equation.

We can use vector equations to describe lines in $\mathbb{R}^n$ using the following formula.

Formula 4. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be a direction vector for line l in $\mathbb{R}^n$, and let $(a_1, a_2, \dots, a_n)$ be an arbitrary point on l . Then

the following vector equation describes l :

$$\mathbf{x}(t) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} t + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Practice Problems

Problem 5 In Explorations ??, ?? and ?? you encountered a set of parametric equations

$$x = 2t - 1$$

$$y = -3t + 6$$

Show that the following parametric equations describe the same line.

$$x = 2t + 1$$

$$y = -3t + 3$$

Problem 6 Find two different parametric representations for a line with slope 3 and y -intercept $(0, -4)$.

Problem 7 Suppose line l has a parametric representation:

$$x_1 = t + 3$$

$$x_2 = t - 4$$

$$x_3 = 3t$$

$$x_4 = 7t + 1$$

Find the missing components for the two vectors below if we know that both vectors can be considered direction vectors for line l .

$$\begin{bmatrix} \dots \\ \dots \\ \dots \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ \dots \\ \dots \\ \dots \end{bmatrix}$$