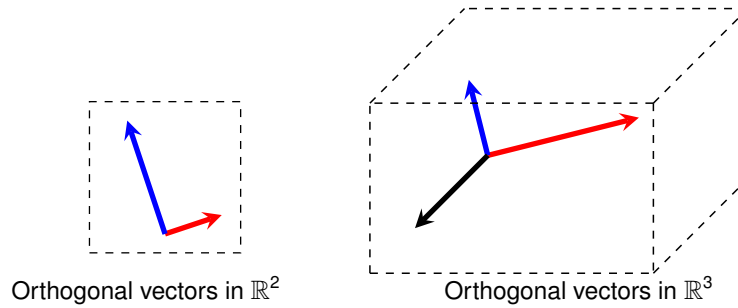
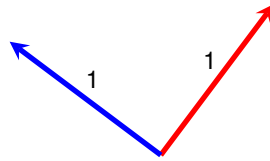

1 Orthogonality and Projections

Orthogonal and Orthonormal Sets

In this section, we examine what it means for vectors (and sets of vectors) to be orthogonal and orthonormal. Recall that two non-zero vectors are orthogonal if their dot product is zero. A collection of non-zero vectors in $\mathbb{R}^n$ is called *orthogonal* if the vectors are pair-wise orthogonal. The diagram below shows two orthogonal vectors in $\mathbb{R}^2$ and three orthogonal vectors in $\mathbb{R}^3$.



If every vector in an orthogonal set of vectors is also a unit vector, then we say that the given set of vectors is *orthonormal*.



An orthonormal set of two vectors

Formally, we can define orthogonal and orthonormal vectors as follows.

Definition 1. Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of nonzero vectors in $\mathbb{R}^n$. Then this set is called an *orthogonal set* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$. Moreover, if $\|\mathbf{v}_i\| = 1$ for $i = 1, \dots, m$ (i.e. each vector in the set is a unit vector), we say the set of vectors is an *orthonormal set*.

An orthogonal set of vectors may not be orthonormal. To convert an orthogonal set to an orthonormal set, we need to divide each vector by its own length.

Definition 2. *Normalizing* an orthogonal set is the process of turning an orthogonal set into an orthonormal set. If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is an orthogonal subset of $\mathbb{R}^n$, then

$$\left\{ \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1, \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2, \dots, \frac{1}{\|\mathbf{v}_k\|} \mathbf{v}_k \right\}$$

is an orthonormal set.

We illustrate this concept in the following example.

Example 1. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal set of vectors but not an orthonormal one. Find the corresponding orthonormal set.

Explanation. One easily verifies that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal set of vectors. On the other hand one can compute that $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \sqrt{2} \neq 1$ and so the set is not orthonormal.

To find a corresponding orthonormal set, we need to normalize each vector.

$$\begin{aligned} \mathbf{q}_1 &= \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{q}_2 &= \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

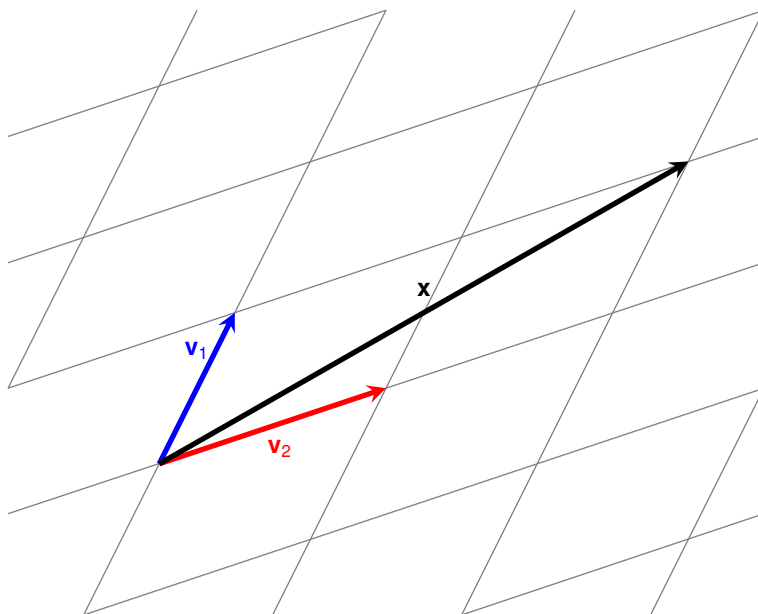
Therefore the corresponding orthonormal set is

$$\{\mathbf{q}_1, \mathbf{q}_2\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

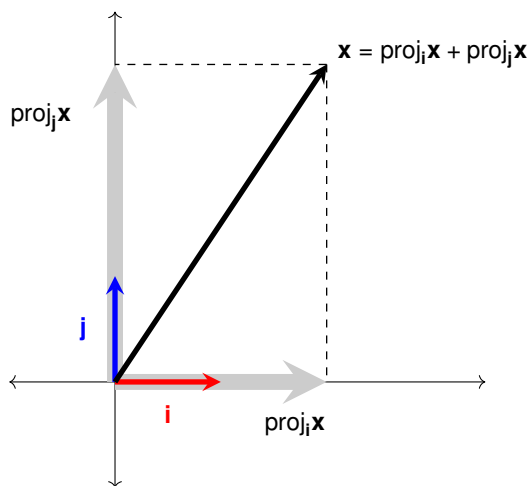
You can verify that this set is orthonormal.

Orthogonal and Orthonormal Bases

Recall that every basis of $\mathbb{R}^n$ (or a subspace W of $\mathbb{R}^n$) imposes a coordinate system on $\mathbb{R}^n$ (or W) that can be used to express any vector of $\mathbb{R}^n$ (or W) as a linear combination of the elements of the basis. For example, vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ impose a coordinate system onto the plane, as shown in the figure below. We readily see that $\mathbf{x}$, contained in the plane, can be written as $\mathbf{x} = \mathbf{v}_1 + 2\mathbf{v}_2$.



Vector $\mathbf{x}$ is visually easy to work with. In general, one way to express an arbitrary vector as a linear combination of the basis vectors is to solve a system of linear equations, which can be costly. One reason we like $\{\mathbf{i}, \mathbf{j}\}$ as a basis of $\mathbb{R}^2$ is because any vector $\mathbf{x}$ of $\mathbb{R}^2$ can be easily expressed as the sum of the orthogonal projections of $\mathbf{x}$ onto the basis vectors $\mathbf{i}$ and $\mathbf{j}$, as shown below.



We can see why an “upright” coordinate system with basis $\{\mathbf{i}, \mathbf{j}\}$ works well. What if we tilt this coordinate system while preserving the orthogonal relationship between the basis vectors? The following exploration allows you to investigate the consequences.

Exploration 1 In the following GeoGebra interactive, vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal (slopes of the lines containing them are negative reciprocals of each other). These vectors are clearly linearly independent and span $\mathbb{R}^2$. Therefore $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of $\mathbb{R}^2$.

Let $\mathbf{x}$ be an arbitrary vector. Orthogonal projections of $\mathbf{x}$ onto $\mathbf{v}_1$ and $\mathbf{v}_2$ are depicted in light grey.

- Use the tip of vector $\mathbf{x}$ to manipulate the vector and convince yourself that $\mathbf{x}$ is always the diagonal of the parallelogram (a rectangle!) determined by the projections.
- Use the tips of $\mathbf{v}_1$ and $\mathbf{v}_2$ to change the basis vectors. What happens when $\mathbf{v}_1$ and $\mathbf{v}_2$ are no longer orthogonal?
- Pick another pair of orthogonal vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Verify that $\mathbf{x}$ is the sum of its projections.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

As you have just discovered in Exploration ??, we can express an arbitrary vector of $\mathbb{R}^2$ as the sum of its projections onto the basis vectors, provided that the basis is orthogonal. It turns out that this result holds for any subspace of $\mathbb{R}^n$, making a basis consisting of orthogonal vectors especially useful.

If an orthogonal set is a basis, we call it an *orthogonal basis*. Similarly, if an orthonormal set is a basis, we call it an *orthonormal basis*.

The following theorem generalizes our observation in Exploration ?? . As you read the statement of the theorem, it will be helpful to recall that the orthogonal projection of vector $\mathbf{x}$ onto a non-zero vector $\mathbf{d}$ is given by

$$\text{proj}_{\mathbf{d}} \mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d} \quad (1)$$

(See [Orthogonal Projections](#).)

Theorem 1. Let W be a subspace of $\mathbb{R}^n$ and suppose $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is an orthogonal basis of W . Then for every $\mathbf{x}$ in W ,

$$\mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} \right) \mathbf{f}_1 + \left(\frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2} \right) \mathbf{f}_2 + \dots + \left(\frac{\mathbf{x} \cdot \mathbf{f}_m}{\|\mathbf{f}_m\|^2} \right) \mathbf{f}_m. \quad (2)$$

Proof We may express $\mathbf{x}$ as a linear combination of the basis elements:

$$\mathbf{x} = c_1 \mathbf{f}_1 + c_2 \mathbf{f}_2 + \dots + c_m \mathbf{f}_m.$$

We claim that $c_i = \frac{\mathbf{x} \cdot \mathbf{f}_i}{\|\mathbf{f}_i\|^2}$ for $i = 1, \dots, m$. To see this, we take the dot product of each side with the vector $\mathbf{f}_i$ and obtain the following.

$$\mathbf{x} \cdot \mathbf{f}_i = (c_1 \mathbf{f}_1 + c_2 \mathbf{f}_2 + \dots + c_m \mathbf{f}_m) \cdot \mathbf{f}_i$$

Our basis is orthogonal, so $\mathbf{f}_j \cdot \mathbf{f}_i = 0$ for all $j \neq i$, which means after we distribute the dot product, only one term will remain on the right-hand side. We have

$$\mathbf{x} \cdot \mathbf{f}_i = c_i \mathbf{f}_i \cdot \mathbf{f}_i$$

We now divide both sides by $\mathbf{f}_i \cdot \mathbf{f}_i = \|\mathbf{f}_i\|^2$, and since our claim holds for $i = 1, \dots, m$, the proof is complete. ■

Theorem ?? shows one important benefit of a basis being orthogonal. With an orthogonal basis it is easy to represent any vector in terms of the basis vectors.

Example 2. Let $\mathbf{f}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{f}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{f}_3 = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$, and let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Notice that $\mathcal{B} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an orthogonal set of vectors, and $\mathcal{B}$ spans $\mathbb{R}^3$. Use this fact to write $\mathbf{x}$ as a linear combination of the vectors of $\mathcal{B}$.

Explanation. We first observe that $\mathcal{B}$ is a linearly independent set of vectors, and so $\mathcal{B}$ is a basis for $\mathbb{R}^3$. Next we apply Theorem ?? to express $\mathbf{x}$ as a linear combination of the vectors of $\mathcal{B}$. We wish to write:

$$\mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} \right) \mathbf{f}_1 + \left(\frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2} \right) \mathbf{f}_2 + \left(\frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2} \right) \mathbf{f}_3.$$

We readily compute:

$$\frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} = \frac{2}{6}, \quad \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2} = \frac{3}{5}, \quad \text{and} \quad \frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2} = \frac{4}{30}.$$

Therefore,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \frac{3}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + \frac{2}{15} \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}.$$

The formula from Theorem ?? is easy to use, and it becomes even easier when our basis is *orthonormal*.

Corollary 1. Let W be a subspace of $\mathbb{R}^n$ and suppose $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m\}$ is an orthonormal basis of W . Then for any $\mathbf{x}$ in W ,

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{q}_1) \mathbf{q}_1 + (\mathbf{x} \cdot \mathbf{q}_2) \mathbf{q}_2 + \dots + (\mathbf{x} \cdot \mathbf{q}_m) \mathbf{q}_m.$$

Proof This is a special case of Theorem ??. Because $\|\mathbf{u}_i\| = 1$ for $i = 1, \dots, m$, the terms are given by

$$\left(\frac{\mathbf{x} \cdot \mathbf{q}_i}{\|\mathbf{q}_i\|^2} \right) \mathbf{q}_i = (\mathbf{x} \cdot \mathbf{q}_i) \mathbf{q}_i.$$

■

Orthogonal Projection onto a Subspace

In the previous section we found that given a subspace W of $\mathbb{R}^n$ with an orthogonal basis $\mathcal{B}$, every vector $\mathbf{x}$ in W can be expressed as the sum of the orthogonal projections of $\mathbf{x}$ onto the elements of $\mathcal{B}$. Note that our premise was that $\mathbf{x}$ is in W . In this section, we look into the meaning of the sum of orthogonal projections of $\mathbf{x}$ onto the elements of an orthogonal basis of W for those vectors $\mathbf{x}$ of $\mathbb{R}^n$ that are *not* in W .

Exploration 2 In the GeoGebra interactive below, W is a plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, in $\mathbb{R}^3$. W is subspace of $\mathbb{R}^3$. In the initial set up, $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Vector $\mathbf{x}$ is not in W .

Use check-boxes to construct the sum of orthogonal projections of $\mathbf{x}$ onto $\mathbf{v}_1$ and $\mathbf{v}_2$. RIGHT-CLICK and DRAG to rotate the image.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

Question 2.1 If moved, return the basis vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ to their default position (set $s_1 = s_2 = 0$) to ensure that they are orthogonal.

- Rotate the image to convince yourself that the perpendiculars dropped from the tip of $\mathbf{x}$ to $\mathbf{v}_1$ and $\mathbf{v}_2$ are indeed perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$ in the diagram. (You'll have to look at it just right to convince yourself of this.) Are both of these perpendiculars also necessarily perpendicular to the plane? (Yes, / No)
- Use sliders x_1 , x_2 and x_3 to manipulate $\mathbf{x}$. Rotate the figure for a better view. What is true about vector $\mathbf{p}$?

Multiple Choice:

1. $\mathbf{p} = \mathbf{x} - (\text{proj}_{\mathbf{v}_1} \mathbf{x} + \text{proj}_{\mathbf{v}_2} \mathbf{x})$.
2. Vector $\mathbf{p}$ is orthogonal to W .
3. All of the above.

- Rotate the figure so that you're looking directly down at the plane. If you're looking at it correctly, you will notice that (1) the parallelogram determined by the projections of $\mathbf{x}$ onto $\mathbf{v}_1$ and $\mathbf{v}_2$ is a rectangle; (2) the sum of projections, $\text{proj}_{\mathbf{v}_1} \mathbf{x} + \text{proj}_{\mathbf{v}_2} \mathbf{x}$, is located directly underneath $\mathbf{x}$, like a shadow at midday.

Question 2.2 Use sliders s_1 and s_2 to manipulate the basis vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ so that they are no longer orthogonal.

- Rotate the figure for a better view. Which of the following is true?

Multiple Choice:

1. $\mathbf{p} = \mathbf{x} - (\text{proj}_{\mathbf{v}_1} \mathbf{x} + \text{proj}_{\mathbf{v}_2} \mathbf{x})$.
2. Vector $\mathbf{p}$ is orthogonal to W .
3. All of the above.

- Rotate your figure so that you're looking directly down at the plane. Which of the following is true?

Multiple Choice:

1. Parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$ is a rectangle.
2. $\text{proj}_{\mathbf{v}_1} \mathbf{x} + \text{proj}_{\mathbf{v}_2} \mathbf{x}$ is located directly underneath $\mathbf{x}$.
3. None of the above.

In Exploration ??, you discovered that given a plane, spanned by orthogonal vectors $\mathbf{v}_1, \mathbf{v}_2$, in $\mathbb{R}^3$, and a vector $\mathbf{x}$, not in the plane, we can interpret the sum of orthogonal projections of $\mathbf{x}$ onto $\mathbf{v}_1$ and $\mathbf{v}_2$ as a “shadow” of $\mathbf{x}$ that lies in the plane directly underneath the vector $\mathbf{x}$. We say that this “shadow” is an *orthogonal projection of $\mathbf{x}$ onto W* . You have also found that if $\mathbf{v}_1, \mathbf{v}_2$ are not orthogonal, the parallelogram representing the sum of the orthogonal projections of $\mathbf{x}$ onto $\mathbf{v}_1$ and $\mathbf{v}_2$ will not be a rectangle. In this case, $\mathbf{x}$ minus this sum will NOT be orthogonal to the plane. It is essential that $\mathbf{v}_1, \mathbf{v}_2$ are orthogonal for $\text{proj}_{\mathbf{v}_1} \mathbf{x} + \text{proj}_{\mathbf{v}_2} \mathbf{x}$ to be considered an orthogonal projection.

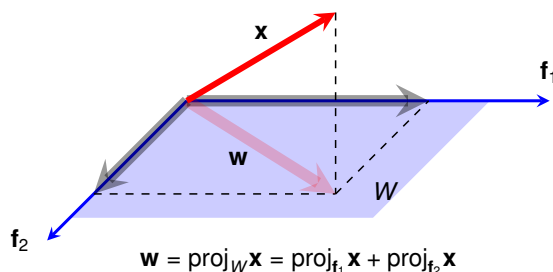
In general, we can define an orthogonal projection of $\mathbf{x}$ in $\mathbb{R}^n$ onto a subspace W of $\mathbb{R}^n$ as the sum of the orthogonal projections of $\mathbf{x}$ onto the elements of an orthogonal basis of W . Definition ?? and the subsequent diagram summarize this discussion.

Definition 3 (Projection onto a Subspace of $\mathbb{R}^n$). Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$. If $\mathbf{x}$ is in $\mathbb{R}^n$, the vector

$$\mathbf{w} = \text{proj}_W(\mathbf{x}) = \text{proj}_{\mathbf{f}_1} \mathbf{x} + \text{proj}_{\mathbf{f}_2} \mathbf{x} + \dots + \text{proj}_{\mathbf{f}_m} \mathbf{x} \quad (3)$$

is called the *orthogonal projection* of $\mathbf{x}$ onto W .

An illustration of Definition ?? for a two-dimensional subspace W with orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ is shown below.



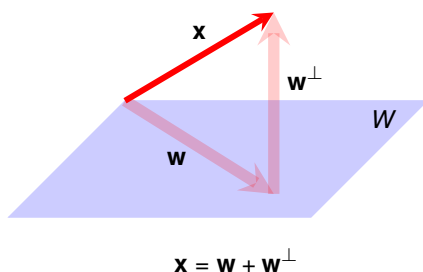
Using equation (??) multiple times, we can also express $\mathbf{w}$ in Definition ?? using the following formula.

Formula 1.

$$\mathbf{w} = \text{proj}_W(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} \mathbf{f}_1 + \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2} \mathbf{f}_2 + \dots + \frac{\mathbf{x} \cdot \mathbf{f}_m}{\|\mathbf{f}_m\|^2} \mathbf{f}_m \quad (4)$$

Orthogonal Decomposition of $\mathbf{x}$

Definition ?? allows us to express $\mathbf{x}$ as the sum of its orthogonal projection, $\mathbf{w} = \text{proj}_W \mathbf{x}$, located in W , and a vector we will call $\mathbf{w}^\perp$ (pronounced “W-perp”), given by $\mathbf{w}^\perp = \mathbf{x} - \mathbf{w}$. This decomposition of $\mathbf{x}$ is shown in the diagram below.



You have already met $\mathbf{w}^\perp$, under the name of $\mathbf{p}$ in Exploration ??, and observed that this vector is orthogonal to W . We will now prove that $\mathbf{w}^\perp$ is orthogonal to every vector in W . This will be accomplished in two steps. First, in Theorem ?? we will prove that $\mathbf{w}^\perp$ is orthogonal to all of the basis elements of W . Next, you will use this result to demonstrate that $\mathbf{w}^\perp$ is orthogonal to every vector in W .

Theorem 2. Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$. Let $\mathbf{x}$ be in $\mathbb{R}^n$, and define $\mathbf{w}^\perp$ as

$$\mathbf{w}^\perp = \mathbf{x} - \text{proj}_W \mathbf{x} = \mathbf{x} - (\text{proj}_{\mathbf{f}_1} \mathbf{x} + \text{proj}_{\mathbf{f}_2} \mathbf{x} + \dots + \text{proj}_{\mathbf{f}_m} \mathbf{x})$$

Then $\mathbf{w}^\perp$ is orthogonal to $\mathbf{f}_i$ for $1 \leq i \leq m$.

Proof We will use Formula ?? to show that $\mathbf{w}^\perp \cdot \mathbf{f}_i = 0$. Recall that $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is an orthogonal basis. Therefore $\mathbf{f}_j \cdot \mathbf{f}_i = 0$ for $i \neq j$. This observation enables us to compute as follows.

$$\begin{aligned} \mathbf{w}^\perp \cdot \mathbf{f}_i &= \left[\mathbf{x} - \left(\frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} \mathbf{f}_1 + \dots + \frac{\mathbf{x} \cdot \mathbf{f}_i}{\|\mathbf{f}_i\|^2} \mathbf{f}_i + \dots + \frac{\mathbf{x} \cdot \mathbf{f}_m}{\|\mathbf{f}_m\|^2} \mathbf{f}_m \right) \right] \cdot \mathbf{f}_i \\ &= \mathbf{x} \cdot \mathbf{f}_i - \frac{\mathbf{x} \cdot \mathbf{f}_i}{\|\mathbf{f}_i\|^2} (\mathbf{f}_i \cdot \mathbf{f}_i) \\ &= \mathbf{x} \cdot \mathbf{f}_i - \frac{\mathbf{x} \cdot \mathbf{f}_i}{\|\mathbf{f}_i\|^2} \|\mathbf{f}_i\|^2 = \mathbf{x} \cdot \mathbf{f}_i - \mathbf{x} \cdot \mathbf{f}_i = 0 \end{aligned}$$

■

We leave the proof of the following Corollary as Practice Problem ??

Corollary 2. Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$. Let $\mathbf{x}$ be in $\mathbb{R}^n$, and define $\mathbf{w}^\perp$ as

$$\mathbf{w}^\perp = \mathbf{x} - \text{proj}_W \mathbf{x} = \mathbf{x} - (\text{proj}_{\mathbf{f}_1} \mathbf{x} + \text{proj}_{\mathbf{f}_2} \mathbf{x} + \dots + \text{proj}_{\mathbf{f}_m} \mathbf{x})$$

Then $\mathbf{w}^\perp$ is orthogonal to every vector in W .

The fact that the decomposition of $\mathbf{x}$ into the sum of $\mathbf{w}$ and $\mathbf{w}^\perp$ is unique is the subject of the Orthogonal Decomposition Theorem which we will prove in [Orthogonal Complements and Decompositions](#).

Throughout this section we have worked with orthogonal bases of subspaces. Does every subspace of $\mathbb{R}^n$ have an orthogonal basis? If so, how do we find one? These questions will be addressed in [Gram-Schmidt Orthogonalization](#).

Practice Problems

Problem 3 *Retry Example ?? using Gaussian elimination.*

Problem 4 Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathbb{R}^n$ and suppose $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\} = \mathbb{R}^n$. Furthermore, suppose that there exists a vector $\mathbf{v} \in \mathbb{R}^n$ for which $\mathbf{v} \cdot \mathbf{x}_j = 0$ for all j , $1 \leq j \leq k$. Show that $\mathbf{v} = \mathbf{0}$.

Problem 5 Let $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 6 \end{bmatrix}$ in $\mathbb{R}^4$, and let $W = \text{span} \left(\begin{bmatrix} 2 \\ 1 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$.

Question 5.1 Compute $\text{proj}_W(\mathbf{x})$.

Answer:

$$\frac{1}{10} \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

Question 5.2 Show that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 1 \\ 2 \end{bmatrix} \right\}$ is another orthogonal basis of W .

Question 5.3 Use the basis you found in the previous part to compute $\text{proj}_W(\mathbf{x})$.

Answer:

$$\frac{1}{70} \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 6 Prove Corollary ??

Text Source

A portion of the text in this section is an adaptation of Section 4.11.1 of Ken Kuttler's [A First Course in Linear Algebra](#). (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 233-238.