
1 SVD Decomposition

We begin this section with an important definition.

Definition 1. Let A be an $m \times n$ matrix. The *singular values* of A are the square roots of the positive eigenvalues of $A^T A$.

Singular Value Decomposition (SVD) can be thought of as a generalization of orthogonal diagonalization of a symmetric matrix to an arbitrary $m \times n$ matrix. This decomposition is the focus of this section.

The following is a useful result that will help when computing the SVD of matrices.

Lemma 1. Let A be an $m \times n$ matrix. Then $A^T A$ and AA^T have the same **nonzero eigenvalues**.

Proof Suppose A is an $m \times n$ matrix, and suppose that λ is a nonzero eigenvalue of $A^T A$. Then there exists a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$(A^T A)\mathbf{x} = \lambda \mathbf{x} \tag{1}$$

Multiplying both sides of this equation by A yields:

$$\begin{aligned} A(A^T A)\mathbf{x} &= A\lambda \mathbf{x} \\ (AA^T)(A\mathbf{x}) &= \lambda(A\mathbf{x}) \end{aligned}$$

Since $\lambda \neq 0$ and $\mathbf{x} \neq 0_n$, $\lambda \mathbf{x} \neq 0_n$, and thus by equation (??), $(A^T A)\mathbf{x} \neq 0_m$; thus $A^T(A\mathbf{x}) \neq 0_m$, implying that $A\mathbf{x} \neq 0_m$.

Therefore $A\mathbf{x}$ is an eigenvector of AA^T corresponding to eigenvalue λ . An analogous argument can be used to show that every nonzero eigenvalue of AA^T is an eigenvalue of $A^T A$, thus completing the proof. ■

Given an $m \times n$ matrix A , we will see how to express A as a product

$$A = U\Sigma V^T$$

where

- U is an $m \times m$ orthogonal matrix whose columns are eigenvectors of AA^T .
- V is an $n \times n$ orthogonal matrix whose columns are eigenvectors of $A^T A$.
- Σ is an $m \times n$ matrix whose only nonzero values lie on its main diagonal, and are the singular values of A .

How can we find such a decomposition? We are aiming to decompose A in the following form:

$$A = U \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

where σ is a block matrix of the form

$$\sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{bmatrix}$$

Thus $A^T = V \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$ and it follows that

$$A^T A = V \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} U^T U \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} V^T = V \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

and so $A^T A V = V \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$. Similarly, $A A^T U = U \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, you would find an orthonormal basis of eigenvectors for $A A^T$ make them the columns of a matrix such that the corresponding eigenvalues are decreasing. This gives U . You could then do the same for $A^T A$ to get V .

We formalize this discussion in the following theorem.

Theorem 1 (Singular Value Decomposition). Let A be an $m \times n$ matrix. Then there exist orthogonal matrices U and V of the appropriate size such that $A = U \Sigma V^T$ where Σ is of the form

$$\Sigma = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}$$

and σ is a block matrix of the form

$$\sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{bmatrix}$$

for the σ_i the singular values of A .

Proof There exists an orthonormal basis, $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ such that $A^T A \mathbf{v}_i = \sigma_i^2 \mathbf{v}_i$ where $\sigma_i^2 > 0$ for $i = 1, \dots, k$, ($\sigma_i > 0$) and equals zero if $i > k$, $A \mathbf{v}_i = \mathbf{0}$ because

$$A \mathbf{v}_i \cdot A \mathbf{v}_i = (A \mathbf{v}_i)^T (A \mathbf{v}_i) = \mathbf{v}_i^T (A^T A \mathbf{v}_i) = \mathbf{0}$$

For $i = 1, \dots, k$, define $\mathbf{u}_i \in \mathbb{R}^m$ by

$$\mathbf{u}_i = \sigma_i^{-1} A \mathbf{v}_i$$

Thus $A \mathbf{v}_i = \sigma_i \mathbf{u}_i$. Now

$$\begin{aligned} \mathbf{u}_i \cdot \mathbf{u}_j &= \sigma_i^{-1} A \mathbf{v}_i \cdot \sigma_j^{-1} A \mathbf{v}_j = \sigma_i^{-1} \mathbf{v}_i^T \sigma_j^{-1} A^T A \mathbf{v}_j \\ &= \sigma_i^{-1} \mathbf{v}_i \cdot \sigma_j^{-1} \sigma_j^2 \mathbf{v}_j = \frac{\sigma_j}{\sigma_i} (\mathbf{v}_i \cdot \mathbf{v}_j) \end{aligned}$$

This means that $\mathbf{u}_i \cdot \mathbf{u}_j = 1$ when $i = j$ and $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ when $i \neq j$. Thus $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthonormal set of vectors in $\mathbb{R}^m$. Also,

$$A A^T \mathbf{u}_i = A A^T \sigma_i^{-1} A \mathbf{v}_i = \sigma_i^{-1} A A^T A \mathbf{v}_i = \sigma_i^{-1} A \sigma_i^2 \mathbf{v}_i = \sigma_i^2 \mathbf{u}_i$$

Now, using Gram-Schmidt, extend $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ to an orthonormal basis for all of $\mathbb{R}^m$, $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ and let

$$U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_m \end{bmatrix}$$

while $V = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$. Thus U is the matrix which has the $\mathbf{u}_i$ as columns and V is defined as the matrix which has the $\mathbf{v}_i$ as columns. Then

$$\begin{aligned} U^T A V &= \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_k^T \\ \vdots \\ \mathbf{u}_m^T \end{bmatrix} A \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_k^T \\ \vdots \\ \mathbf{u}_m^T \end{bmatrix} \begin{bmatrix} \sigma_1 \mathbf{u}_1 & \cdots & \sigma_k \mathbf{u}_k & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where σ is given in the statement of the theorem. ■

The SVD has as an immediate corollary which is given in the following interesting result.

Corollary 1. Let A be an $m \times n$ matrix. Then the rank of A (or of A^T) equals the number of singular values.

Let's compute the SVD of a simple matrix.

Example 1. Let $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. Find the SVD of A .

Explanation. To begin, we compute AA^T and $A^T A$.

$$AA^T = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 5 \\ 5 & 11 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}$$

Since AA^T is 2×2 while $A^T A$ is 3×3 , and AA^T and $A^T A$ have the same *nonzero* eigenvalues (by Lemma ??), we compute the characteristic polynomial $c_{AA^T}(x)$ (because it is easier to compute than $c_{A^T A}(x)$).

$$\begin{aligned} c_{AA^T}(z) &= \det(zI - AA^T) = \det \begin{bmatrix} z - 11 & -5 \\ -5 & z - 11 \end{bmatrix} \\ &= (z - 11)^2 - 25 \\ &= z^2 - 22z + 121 - 25 \\ &= z^2 - 22z + 96 \\ &= (z - 16)(z - 6) \end{aligned}$$

Therefore, the eigenvalues of AA^T are $\lambda_1 = 16$ and $\lambda_2 = 6$.

The eigenvalues of $A^T A$ are $\lambda_1 = 16$, $\lambda_2 = 6$, and $\lambda_3 = 0$, and the singular values of A are $\sigma_1 = \sqrt{16} = 4$ and $\sigma_2 = \sqrt{6}$. By convention, we list the eigenvalues (and corresponding singular values) in non increasing order (i.e., from largest to smallest).

To find the matrix V :

To construct the matrix V we need to find eigenvectors for $A^T A$. Since the eigenvalues of AA^T are distinct, the corresponding eigenvectors are orthogonal, and we need only normalize them.

$\lambda_1 = 16$: solve $(16I - A^T A)\mathbf{x}_1 = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} 6 & -2 & -6 & 0 \\ -2 & 14 & 2 & 0 \\ -6 & 2 & 6 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right], \text{ so } \mathbf{x}_1 = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, t \in \mathbb{R}.$$

$\lambda_2 = 6$: solve $(6I - A^T A)\mathbf{x}_2 = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} -4 & -2 & -6 & 0 \\ -2 & 4 & 2 & 0 \\ -6 & 2 & -4 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right], \text{ so } \mathbf{x}_2 = \begin{bmatrix} -s \\ -s \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, s \in \mathbb{R}.$$

$\lambda_3 = 0$: solve $(-A^T A)\mathbf{x}_3 = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} -10 & -2 & -6 & 0 \\ -2 & -2 & 2 & 0 \\ -6 & 2 & -10 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right], \text{ so } \mathbf{x}_3 = \begin{bmatrix} -r \\ 2r \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, r \in \mathbb{R}.$$

Let

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Then

$$V = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & -\sqrt{2} & -1 \\ 0 & -\sqrt{2} & 2 \\ \sqrt{3} & \sqrt{2} & 1 \end{bmatrix}$$

Also,

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix}$$

and we use A , V^T , and Σ to find U . Since V is orthogonal and $A = U\Sigma V^T$, it follows that $AV = U\Sigma$. Let $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$, and let $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the two columns of U . Then we have

$$\begin{aligned} A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \Sigma \\ \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & A\mathbf{v}_3 \end{bmatrix} &= \begin{bmatrix} \sigma_1 \mathbf{u}_1 + 0\mathbf{u}_2 & 0\mathbf{u}_1 + \sigma_2 \mathbf{u}_2 & 0\mathbf{u}_1 + 0\mathbf{u}_2 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1 \mathbf{u}_1 & \sigma_2 \mathbf{u}_2 & 0 \end{bmatrix} \end{aligned}$$

which implies that $A\mathbf{v}_1 = \sigma_1\mathbf{u}_1 = 4\mathbf{u}_1$ and $A\mathbf{v}_2 = \sigma_2\mathbf{u}_2 = \sqrt{6}\mathbf{u}_2$. Thus,

$$\mathbf{u}_1 = \frac{1}{4}A\mathbf{v}_1 = \frac{1}{4} \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{4\sqrt{2}} \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\mathbf{u}_2 = \frac{1}{\sqrt{6}}A\mathbf{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Therefore,

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix} \left(\frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & 0 & \sqrt{3} \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ -1 & 2 & 1 \end{bmatrix} \right) \end{aligned}$$

Here is another example.

Example 2. Find an SVD for $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

Explanation. Since A is 3×1 , $A^T A$ is a 1×1 matrix whose eigenvalues are easier to find than the eigenvalues of the 3×3 matrix AA^T .

$$A^T A = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}$$

Thus $A^T A$ has eigenvalue $\lambda_1 = 9$, and the eigenvalues of AA^T are $\lambda_1 = 9$, $\lambda_2 = 0$, and $\lambda_3 = 0$. Furthermore, A has only one singular value, $\sigma_1 = 3$.

To find the matrix V : To do so we find an eigenvector for $A^T A$ and normalize it. In this case, finding a unit eigenvector is trivial: $\mathbf{v}_1 = \begin{bmatrix} 1 \end{bmatrix}$, and

$$V = \begin{bmatrix} 1 \end{bmatrix}$$

Also, $\Sigma = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, and we use A , V^T , and Σ to find U .

Now $AV = U\Sigma$, with $V = \begin{bmatrix} \mathbf{v}_1 \end{bmatrix}$, and $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$, where $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are the columns of U . Thus

$$\begin{aligned} A \begin{bmatrix} \mathbf{v}_1 \end{bmatrix} &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} \Sigma \\ \begin{bmatrix} A\mathbf{v}_1 \end{bmatrix} &= \begin{bmatrix} \sigma_1 \mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1 \mathbf{u}_1 \end{bmatrix} \end{aligned}$$

This gives us $A\mathbf{v}_1 = \sigma_1 \mathbf{u}_1 = 3\mathbf{u}_1$, so

$$\mathbf{u}_1 = \frac{1}{3}A\mathbf{v}_1 = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

The vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ are eigenvectors of AA^T corresponding to the eigenvalue $\lambda_2 = \lambda_3 = 0$. Instead of solving the system $(0I - AA^T)\mathbf{x} = 0$ and then using the Gram-Schmidt process on the resulting set of two basic eigenvectors, the following approach may be used.

Find vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ by first extending $\{\mathbf{u}_1\}$ to a basis of $\mathbb{R}^3$, then using the Gram-Schmidt algorithm to orthogonalize the basis, and finally normalizing the vectors.

Starting with $\{3\mathbf{u}_1\}$ instead of $\{\mathbf{u}_1\}$ makes the arithmetic a bit easier. It is easy to verify that

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis of $\mathbb{R}^3$. Set

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and apply the Gram-Schmidt algorithm to $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. This gives us

$$\mathbf{f}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{f}_2 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{f}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Therefore,

$$\mathbf{u}_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

and

$$U = \begin{bmatrix} -\frac{1}{3} & \frac{4}{\sqrt{18}} & 0 \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Finally,

$$A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{\sqrt{18}} & 0 \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

Consider another example.

Example 3. Find an SVD for the matrix

$$A = \begin{bmatrix} \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \end{bmatrix}$$

Explanation. First consider $A^T A$

$$\begin{bmatrix} \frac{16}{5} & \frac{32}{5} & 0 \\ \frac{32}{5} & \frac{64}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

What are some eigenvalues and eigenvectors? Some computing shows that the eigenvalues are 0 and 16. Furthermore, we can find a basis for each eigenspace.

$$\mathcal{S}_0 = \text{span} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \\ 0 \end{bmatrix} \right), \quad \mathcal{S}_{16} = \text{span} \left(\begin{bmatrix} \frac{1}{5}\sqrt{5} \\ \frac{2}{5}\sqrt{5} \\ 0 \end{bmatrix} \right)$$

Thus the matrix V is given by

$$V = \begin{bmatrix} \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next consider AA^T

$$\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

Eigenvalues are 0 and 16, and eigenspaces are

$$\mathcal{S}_0 = \text{span} \left(\begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix} \right), \quad \mathcal{S}_{16} = \text{span} \left(\begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix} \right)$$

Thus you can let U be given by

$$U = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

Let's check this. $U^T AV =$

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \\ \frac{2}{5} & \frac{4}{5} & 0 \\ \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This illustrates that if you have a good way to find the eigenvectors and eigenvalues for a Hermitian matrix which has nonnegative eigenvalues, then you also have a good way to find the SVD of an arbitrary matrix.

Practice Problems

Problem 1 If A is a square matrix, show that $|\det(A)|$ is the product of the singular values of A .

Problem 2 Find an SVD for each of the following matrices.

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}$$

Problem 3 Find an SVD for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Text Source

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