
1 Review for the Final Exam

Exercise 1 Suppose A is a 3×5 matrix, and $\text{rank}(A) = 2$.

1. Which of the following are possibilities for a non-zero vector $\mathbf{b}$? (Check all that apply.)

Select All Correct Answers:

- (i) $A\mathbf{x} = \mathbf{b}$ has a unique solution.
- (ii) $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.
- (iii) $A\mathbf{x} = \mathbf{b}$ has no solutions.

2. Find the following dimensions.

$$\dim(\text{null}(A)) = \dots\dots, \quad \dim(\text{col}(A)) = \dots\dots, \quad \dim(\text{row}(A)) = \dots\dots$$

3. Let T be a linear transformation induced by A . What is true about T ? (Check all that apply.)

Select All Correct Answers:

- (i) $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$
- (ii) $T : \mathbb{R}^3 \rightarrow \mathbb{R}^5$
- (iii) $\mathbf{0}$ is the ONLY vector that maps to $\mathbf{0}$.
- (iv) Infinitely many vectors map to $\mathbf{0}$.
- (v) $\dim(\text{im}(T)) = 2$
- (vi) $\dim(\text{im}(T)) = 3$
- (vii) $\dim(\text{im}(T)) = 5$

Exercise 2 Suppose A is a 7×2 matrix, and $\text{rank}(A) = 2$.

1. Which of the following are possibilities for a non-zero vector $\mathbf{b}$? (Check all that apply.)

Select All Correct Answers:

- (i) $A\mathbf{x} = \mathbf{b}$ has a unique solution.
- (ii) $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.
- (iii) $A\mathbf{x} = \mathbf{b}$ has no solutions.

2. Which of the following is true about the columns of A ?

Multiple Choice:

- (i) Columns of A are linearly independent.

(ii) Columns of A are linearly dependent.

(iii) Not enough information is given to determine linear dependence/independence of the columns of A .

3. Let T be a linear transformation induced by A . What is true about T ? (Check all that apply.)

Select All Correct Answers:

(i) $T : \mathbb{R}^7 \rightarrow \mathbb{R}^2$

(ii) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^7$

(iii) $\mathbf{0}$ is the ONLY vector that maps to $\mathbf{0}$.

(iv) Infinitely many vectors map to $\mathbf{0}$.

(v) $\dim(\text{im}(T)) = 2$

(vi) $\dim(\text{im}(T)) = 7$

Exercise 3 You may use technology for this problem. Let

$$S = \text{span} \left(\begin{bmatrix} 1 \\ -3 \\ 14 \end{bmatrix}, \begin{bmatrix} 17 \\ 0 \\ 16 \end{bmatrix}, \begin{bmatrix} 10 \\ 4 \\ -8 \end{bmatrix} \right)$$

1. Describe S .

Multiple Choice:

(i) S is ALL of $\mathbb{R}^3$.

(ii) S is a plane in $\mathbb{R}^3$.

(iii) S is a line in $\mathbb{R}^3$.

2. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \\ 14 \end{bmatrix}, \begin{bmatrix} 17 \\ 0 \\ 16 \end{bmatrix}, \begin{bmatrix} 10 \\ 4 \\ -8 \end{bmatrix} \right\}$. Is $\mathcal{B}$ a basis for S ?

Multiple Choice:

(i) Yes

(ii) No

Exercise 4 Let A be a 3×3 matrix. Suppose $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. What do you know (or can deduce) from this information? (Check all that apply.)

Select All Correct Answers:

-
1. $\det(A) = 0$
 2. I can find the row space of A .
 3. I can find the column space of A .
 4. Columns of A are linearly dependent.
 5. Rows of A are linearly dependent.
 6. A is not invertible.
 7. The null space of A has infinitely many elements.
 8. 0 is an eigenvalue of A .

Exercise 5 Find the standard matrix, A , of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$$

$$A = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

Exercise 6 Find the determinant of A if

$$A = \begin{bmatrix} 1 & 3 & 4 & 1 \\ k & 0 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

where k is a constant.

$$\det(A) = \dots$$

For what value of k is A singular?

$$k = \dots$$

Exercise 7 Use technology to find the line of best fit.

Desmos link: <https://www.desmos.com/calculator/xfoeppxjbn>

Enter your computed coefficients to three decimal places.

$$y = \dots X + \dots$$

Plot your your line of best fit in the Desmos interactive above.

Exercise 8 True or False? If False, you should come up with a counterexample.

Suppose A and B are $n \times n$ matrices ($n \geq 2$).

1. $\det(AB) = \det A \det B$

Multiple Choice:

- (i) True
- (ii) False

2. If $\det A = \frac{1}{\det B}$, then $A = B^{-1}$.

Multiple Choice:

- (i) True
- (ii) False

3. $\det(A + B) = \det A + \det B$.

Multiple Choice:

- (i) True
- (ii) False

4. If $A = 2B$, then $\det A = 2 \det B$.

Multiple Choice:

- (i) True
- (ii) False

5. If $A = -B$, then $\det A = -\det B$ when n is odd, and $\det A = \det B$ when n is even.

Multiple Choice:

- (i) True
- (ii) False

Exercise 9 Let B be a parallelepiped determined by vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$. Suppose the volume of B is 10 cubic units. Let B' be a parallelepiped determined by $-\mathbf{u}$, $2\mathbf{v}$, and $\mathbf{w} + \mathbf{u}$. Find the volume of B' .

volume of $B' = \dots \dots$ cubic units

Exercise 10 Find eigenvalues of $A = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$.

Eigenvalues (in increasing order): $\lambda_1 = \dots$, $\lambda_2 = \dots$

Compute a basis for the eigenspace associated with each of these eigenvalues.

A basis for S_{λ_1} : $\left\{ \begin{bmatrix} 1 \\ \dots \end{bmatrix} \right\}$

A basis for S_{λ_2} : $\left\{ \begin{bmatrix} 4 \\ \dots \end{bmatrix} \right\}$

Exercise 11 Let $A = \begin{bmatrix} 2 & 3 & -3 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$.

Each of these matrices has the same characteristic polynomial, $(2 - \lambda)(1 + \lambda)^2$. Which of the following is true? (Justify your answer.)

Multiple Choice:

1. A is diagonalizable but B is not diagonalizable.
2. A is not diagonalizable but B is diagonalizable.
3. Both A and B are diagonalizable.
4. Neither A nor B is diagonalizable.

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Problem 3.3.26.

Exercise 12 For square matrices A and B. True or False? If False, you should come up with a counterexample. If True, can you give a proof?

1. If λ is an eigenvalue of A, then $\lambda - c$ is an eigenvalue of $A - cI$.

Multiple Choice:

- (i) True
- (ii) False

2. If λ is an eigenvalue of A, then 2λ is an eigenvalue of $2A$.

Multiple Choice:

- (i) True
- (ii) False

Exercise 13 If $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 7 \\ 0 & 7 & 3 \end{bmatrix}$, find an orthogonal matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$.

$$Q = \begin{bmatrix} 1 & \dots\dots & \dots\dots \\ \dots\dots & 1/\sqrt{2} & 1/\sqrt{2} \\ \dots\dots & 1/\sqrt{2} & \dots\dots \end{bmatrix}$$

$$D = \begin{bmatrix} \dots\dots & 0 & 0 \\ 0 & \dots\dots & 0 \\ 0 & 0 & \dots\dots \end{bmatrix}$$

Verify that the columns of Q are orthonormal. Are the rows of Q orthonormal?

Exercise 14 Each set V given below is a subset of $\mathbb{M}_{2,2}$ with the usual matrix operations. Is V a subspace of $\mathbb{M}_{2,2}$?

1. V is the set of 2×2 matrices with zero determinant.

Multiple Choice:

- (i) V is a vector space.
- (ii) V is not a vector space.

2. V is the set of all 2×2 matrices whose entries sum to 0.

Multiple Choice:

- (i) V is a vector space.
- (ii) V is not a vector space.

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 6.1.2.

Exercise 15 Describe the following subspace of $\mathbb{M}_{3,3}$.

$$\text{span} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

The subspace consists of:

Multiple Choice:

1. Matrices A of $\mathbb{M}_{3,3}$ such that $A^T = A$.
2. All invertible matrices of $\mathbb{M}_{3,3}$.
3. Upper and lower triangular matrices of $\mathbb{M}_{3,3}$.

4. All of $\mathbb{M}_{3,3}$.

Exercise 16 Find the matrix of the linear transformation $T : \mathbb{M}_{2,2} \rightarrow \mathbb{M}_{2,2}$, given by $T(A) = A^T$, if the basis of the domain ($\mathcal{B}$) and the basis of the codomain ($\mathcal{D}$) are given by

$$\mathcal{B} = \mathcal{D} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 9.1.3 (b).

Exercise 17 Find the coordinates of $\mathbf{v} = ax^2 + bx + c$ of $\mathbb{P}^2$ with respect to the ordered basis $\mathcal{B} = \{x^2, x + 1, x + 2\}$.

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 9.1.1 (b).

Exercise 18 Suppose $T : \mathbb{M}_{2,2} \rightarrow \mathbb{R}$ is a linear transformation such that

$$T \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = 3$$

$$T \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = -1$$

$$T \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right) = 0$$

$$T \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

Find $T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$

$$T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \dots$$

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 7.1.4 (d).

Exercise 19 Suppose $T : \mathbb{P}^2 \rightarrow \mathbb{P}^3$ is a linear transformation such that

$$T(x^2) = x^3, \quad T(x + 1) = 0, \quad T(x - 1) = x$$

Find $T(x^2 + x + 1)$.

$$T(x^2 + x + 1) = \dots$$

[Nicholson] W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2021, Open Edition, Exercises 7.1.4 (c).