
1 Solved Problems

Solved Problems for Chapter 10

Problem 1 Let M, N be subspaces of a vector space V and consider $M + N$ defined as the set of all $m + n$ where $m \in M$ and $n \in N$. Show that $M + N$ is a subspace of V .

Click the arrow to see answer. We need to show that $M + N$ is closed under addition and scalar multiplication. (See Theorem ??.) Suppose $\mathbf{x}$ and $\mathbf{y}$ are in $M + N$. Then $\mathbf{x} = m_1 + n_1$ and $\mathbf{y} = m_2 + n_2$, where m_i is in M and n_i is in N . To see that $M + N$ is closed under scalar multiplication, consider $a\mathbf{x} = a(m_1 + n_1) = am_1 + an_1$. Since am_1 is in M , and an_1 is in N , $a\mathbf{x}$ is in $M + N$. Next, we show that $M + N$ is closed under vector addition. $\mathbf{x} + \mathbf{y} = (m_1 + m_2) + (n_1 + n_2)$. Since $(m_1 + m_2)$ is an element of M , and $(n_1 + n_2)$ is an element of N we have closure under addition.

Problem 2 Let M, N be subspaces of a vector space V . Then $M \cap N$ consists of all vectors which are in both M and N . Show that $M \cap N$ is a subspace of V .

Click the arrow to see answer. If a vector is in both subspaces M and N , then its scalar multiple must also be in both. Same is true about the sum of two elements of $M \cap N$.

Problem 3 Let M, N be subspaces of a vector space $\mathbb{R}^2$. Then $N \cup M$ consists of all vectors which are in either M or N . Show that $N \cup M$ is not necessarily a subspace of $\mathbb{R}^2$ by giving an example where $N \cup M$ fails to be a subspace.

Click the arrow to see answer. As our example, let M to be the first quadrant, and let N be the third quadrant.

Problem 4 Define $T : \mathbb{P}^2 \rightarrow \mathbb{P}^3$ by $T(p(x)) = xp(x)$.

1. Find $T(-x^2 + 2x - 4)$.
2. Is T a linear transformation? If so, prove it. If not, give a counterexample.

Click the arrow to see answer.

1. $T(-x^2 + 2x - 4) = -x^3 + 2x^2 - 4x$
2. $T(ap(x) + bq(x)) = x(ap(x) + bq(x)) = axp(x) + bxq(x) = aT(p(x)) + bT(p(x))$. T is a linear transformation.

Problem 5 Let $p(x) = 4x^2 - x$. Is $p(x)$ in $\text{span}(x^2 + x, x^2 - 1, -x + 2)$? Consider the question in two different ways, then compare your work for each approach.

1. Do this directly using the definition of span (Definition ??).
2. Do this using isomorphisms.

Click the arrow to see answer.

1. Apply the definition of span directly. We are looking for coefficients a , b and c such that

$$ax^2 + ax + bx^2 - b - cx + 2c = 4x^2 - x$$

Collecting like terms on the left and setting their coefficients equal to their counterparts on the right gives us the following system of equations.

$$a + b = 4$$

$$a - c = -1$$

$$-b + 2c = 0$$

This gives rise to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 1 & 0 & -1 & -1 \\ 0 & -1 & 2 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 10/3 \\ 0 & 0 & 1 & 5/3 \end{array} \right]$$

This shows that our system has a unique solution and gives us the specific coefficients to express $p(x)$ as a linear combination of the vectors in the given set.

2. We can also look at this problem in light of isomorphisms. Let's start by mapping

$$x^2 + x \mapsto \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad x^2 - 1 \mapsto \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad -x + 2 \mapsto \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

(Why is this an isomorphism?)

Is the vector $\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$ in the span of the three vectors above? Set up an augmented matrix to answer this question. Compare this matrix to the matrix in part (a).

Problem 6 Let $p(x) = -x^2 + x + 2$. Is $p(x)$ in $\text{span}(x^2 + x + 1, 2x^2 + x)$?

Click the arrow to see answer. $p(x)$ is not in the span of the given vectors. The system

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{array} \right]$$

is infeasible.

Problem 7 Consider the vector space of polynomials of degree at most 2, $\mathbb{P}_2$. Determine whether the following is a basis for $\mathbb{P}_2$.

$$\{x^2 + x + 1, 2x^2 + 2x + 1, x + 1\}$$

Click the arrow to see answer. There is an isomorphism from $\mathbb{R}^3$ to $\mathbb{P}_2$, defined as follows:

$$T(\mathbf{e}_1) = 1, \quad T(\mathbf{e}_2) = x, \quad T(\mathbf{e}_3) = x^2$$

Thus,

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = x^2 + x + 1, \quad T\left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right) = 2x^2 + 2x + 1, \quad T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = 1 + x$$

It follows that if

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$, then the polynomials will be a basis for $\mathbb{P}_2$ because they will be independent. Recall that an isomorphism takes a linearly independent set to a linearly independent set. Also, since T is an isomorphism, it preserves all linear relations.

Problem 8 Find a basis in $\mathbb{P}_2$ for the subspace

$$\text{span}(1 + x + x^2, 1 + 2x, 1 + 5x - 3x^2)$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Click the arrow to see answer. This is the situation in which you have a spanning set and you want to cut it down to form a linearly independent set which is also a spanning set. Use the same isomorphism as above. Since T is an isomorphism, it preserves all linear relations so if such can be found in $\mathbb{R}^3$, the same linear relations will be present in $\mathbb{P}_2$.

Problem 9 Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ as follows.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$

Show that T is one to one.

Click the arrow to see answer.

We need to show that $T(\mathbf{v}) = T(\mathbf{w})$ implies that $\mathbf{v} = \mathbf{w}$. Suppose

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Performing matrix-vector multiplication on both sides shows that $v_1 = w_1$, and $v_2 = w_2$. Therefore, $\mathbf{v} = \mathbf{w}$.

Problem 10 Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ as follows.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$

Is T onto?

Click the arrow to see answer.

T is not onto. One way to show this is by finding a counter-example. We need a vector in $\mathbb{R}^3$ that is not an image of any element of $\mathbb{R}^2$ under T . Try $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$.

Problem 11 Find the coordinates of $\mathbf{v}$ with respect to the ordered basis $\mathcal{B}$ of $\mathbb{M}_{2,2}$.

$$\mathbf{v} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Click the arrow to see answer.

We need to find coefficients a , b , c , and d such that

$$a \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

This leads to the following system of equations:

$$a + b + d = 1$$

$$a = 2$$

$$b + c = -1$$

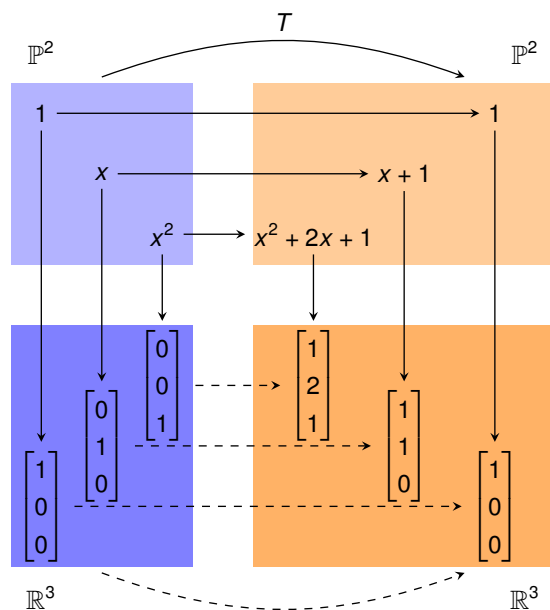
$$c + d = 0$$

We get $a = 2$, $b = -1$, $c = 0$, $d = 0$. Therefore $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$.

Problem 12 Define $T : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ by $T(p(x)) = p(x+1)$. (Verify that T is a linear transformation.) Find the matrix of T if the basis for the domain and the codomain is $\{1, x, x^2\}$.

Click the arrow to see answer.

We start with a diagram:



Based on where the standard unit vectors of $\mathbb{R}^2$ map to, we have the following matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Bibliography

Some of the problems come from the end of Chapter 9 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 469–535.

from Section 10.1 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, pp. 501.