
1 Solved Problems

Solved Problems for Chapter 3

Problem 1 If possible, express the vector

$$\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}$$

as a linear combination of the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Click the arrow to see answer. We start with a linear combination equation $a \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}$. This gives us the following:

$$\left[\begin{array}{cc|c} 3 & 2 & 4 \\ 1 & -2 & 4 \\ -1 & 1 & -3 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right]$$

This gives us $\mathbf{v} = 2\mathbf{u}_1 - \mathbf{u}_2$.

Problem 2 Decide whether

$$\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

is a linear combination of the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Click the arrow to see answer.

$$\left[\begin{array}{cc|c} 3 & 2 & 4 \\ 1 & -2 & 4 \\ -1 & 1 & 4 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

The last row indicates that there is no solution to the linear combination equation $a\mathbf{u}_1 + b\mathbf{u}_2 = \mathbf{v}$. Therefore, $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

Problem 3 Determine whether the following set of vectors is linearly independent.

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

Click the arrow for answer.

The linear combination equation $a \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \mathbf{0}$ has only the trivial solution. We can see this as follows.

$$\text{rref} \left(\begin{bmatrix} 2 & 1 & 1 \\ -3 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore the given set of vectors is linearly independent.

Problem 4 Determine whether the following set of vectors is linearly independent.

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

Click the arrow for answer.

The linear combination equation $a \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} + b \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \mathbf{0}$ has a non-trivial solution. We can see this as follows.

$$\text{rref} \left(\begin{bmatrix} 2 & 4 & 1 \\ -3 & -1 & 1 \\ 0 & 8 & 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the given set of vectors is linearly dependent.

Problem 5 Suppose $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is a set of vectors from $\mathbb{R}^n$. Show that $\mathbf{0}$ is in $\text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$.

Click the arrow to see answer. Let all of the coefficients in the linear combination be zero: $\sum_{i=1}^k 0\mathbf{x}_i = \mathbf{0}$

Problem 6 True or false?

Click the arrow to see each answer.

1. The span of any one vector in $\mathbb{R}^n$ is a line.

This is technically false because the span of the zero vector is a point. For all non-zero vectors this statement is true.

2. The span of two non-zero vectors in $\mathbb{R}^n$ is a plane.

This is false. If two vectors are scalar multiples of each other (i.e. they are linearly dependent), then their span is a line, not a plane.

3. Given two linearly independent vectors, $\mathbf{v}$ and $\mathbf{w}$, let $\mathbf{n}$ be a normal vector to the plane spanned by $\mathbf{v}$ and $\mathbf{w}$. The set $\{\mathbf{v}, \mathbf{w}, \mathbf{n}\}$ is linearly independent.

This is true. A normal to the plane is a vector perpendicular to the plane. Only vectors that lie in the plane spanned by $\mathbf{v}$ and $\mathbf{w}$ can be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. Since $\mathbf{n}$ is not in the plane, it cannot be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. Therefore the set $\{\mathbf{v}, \mathbf{w}, \mathbf{n}\}$ is linearly independent.

Bibliography

Some of the Review Exercises come from the end of Chapter 4 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 151–152.