
1 Solved Problems

Solved Problems for Chapter 7

Problem 1 Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ -2 & 5 & 1 \end{bmatrix}$. When doing cofactor expansion along the top row, we encounter three minor matrices. What are they?

Click the arrow to see answer.

$$\begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix}$$

Problem 2 Find the following determinant by (a) expanding along the first row, (b) second column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

Click the arrow to see answer.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 6$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 6$$

Problem 3 Compute the determinant by co-factor expansion. Pick the easiest row or column to use.

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix}$$

Click the arrow to see answer. Expand along the third row.

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -2 \left(1 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \right) = -2(2) = -4$$

Problem 4 An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Click the arrow to see answer. It does not change the determinant. This was just taking the transpose.

Problem 5 An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Click the arrow to see answer. In this case two rows were switched and so the resulting determinant is -1 times the first.

Problem 6 An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$$

Click the arrow to see answer. The determinant is unchanged. It was just the first row added to the second.

Problem 7 An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$$

Click the arrow to see answer. The second row was multiplied by 2 so the determinant of the result is 2 times the original determinant.

Problem 8 An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Click the arrow to see answer. In this case the two columns were switched so the determinant of the second is -1 times the determinant of the first.

Problem 9 Let A be an $n \times n$ matrix and suppose there are $n - 1$ rows such that all rows are linear combinations of these $n - 1$ rows. Show $\det(A) = 0$.

Click the arrow to see answer. If the determinant is nonzero, then it will remain nonzero with row operations applied to the matrix. In this case, you can obtain a row of zeros by doing row operations. Thus the determinant must be zero.

Problem 10 Construct 2×2 matrices A and B to illustrate the property that $\det A \det B = \det(AB)$.

Click the arrow to see answer.

$$\det \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix} \right) = -8$$

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \det \begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix} = -2 \times 4 = -8$$

Problem 11 An $n \times n$ matrix is called nilpotent if for some positive integer, k it follows $A^k = O$. If A is a nilpotent matrix and k is the smallest possible integer such that $A^k = O$, what are the possible values of $\det(A)$?

Click the arrow to see answer. $\det A = 0$ because $0 = \det(O) = \det(A^k) = (\det(A))^k$.

Problem 12 A matrix is said to be orthogonal if $A^T A = I$. Thus the inverse of an orthogonal matrix is just its transpose. What are the possible values of $\det(A)$ if A is an orthogonal matrix?

Click the arrow to see answer. You would need $\det(AA^T) = \det(A) \det(A^T) = \det(A)^2 = 1$ and so $\det(A) = 1$, or -1 .

Problem 13 Let A and B be two $n \times n$ matrices. We say that A is **similar** to B and write $A \sim B$ provided that there exists an invertible matrix P such that $A = P^{-1}BP$. Show that if $A \sim B$, then $\det(A) = \det(B)$.

Click the arrow to see answer. $\det(A) = \det(S^{-1}BS) = \det(S^{-1}) \det(B) \det(S) = \det(B) \det(S^{-1}S) = \det(B)$.

Problem 14 Determine whether each statement is true or false. If true, provide a proof. If false, provide a counter example.

1. If any two columns of a square matrix are equal, then the determinant of the matrix equals zero.
2. If A^{-1} exists then $\det(A^{-1}) = \det(A)^{-1}$.
3. If A is a real $n \times n$ matrix, then $\det(A^T A) \geq 0$.
4. If $AX = 0$ for some $X \neq 0$, then $\det(A) = 0$.

Click the arrow to see answer. All of the statements are true.

1. $\det(A) = \det(A^T)$. If two columns of A are equal, then two rows of A^T are equal. Applying row operations to A^T will produce a matrix with a row of zeros.
2. Suppose A is invertible. Then

$$1 = \det(I) = \det(AA^{-1}) = \det(A) \det(A^{-1})$$

Divide both right and left side by $\det(A)$ to obtain the result.
3. This follows from the fact that $\det(A) = \det(A^T)$.
4. If $AX = 0$ for some $X \neq 0$ then there is a non-trivial linear combination of the columns of A that is equal to 0 . This means that the columns of A are linearly dependent. This implies that the rows are also linearly dependent (why?). Applying elementary row operations will lead us to a row of zeros.

Problem 15 Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non-zero.

Click the arrow to see answer.

$$\det \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} = 2$$

and so it has an inverse. The inverse turns out to equal

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & -\frac{2}{1} & 1 \\ \frac{2}{1} & -\frac{2}{2} & 0 \end{bmatrix}$$

Bibliography

Some of the problems come from the end of Chapter 3 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 272–315.