
1 Solved Problems

Solved Problems for Chapter 8

Problem 1 If A is an invertible $n \times n$ matrix, compare the eigenvalues of A and A^{-1} . More generally, for m an arbitrary integer, compare the eigenvalues of A and A^m .

Click the arrow to see answer. $A^m X = \lambda^m X$ for any integer. In the case of -1 , $A^{-1} \lambda X = A A^{-1} X = X$ so $A^{-1} X = \lambda^{-1} X$. Thus the eigenvalues of A^{-1} are just λ^{-1} where λ is an eigenvalue of A .

Problem 2 If A is an $n \times n$ matrix and c is a nonzero constant, compare the eigenvalues of A and cA .

Click the arrow to see answer. Say $AX = \lambda X$. Then $cAX = c\lambda X$ and so the eigenvalues of cA are just $c\lambda$ where λ is an eigenvalue of A .

Problem 3 Let A, B be invertible $n \times n$ matrices which commute. That is, $AB = BA$. Suppose X is an eigenvector of B . Show that then AX must also be an eigenvector for B .

Click the arrow to see answer. $BAX = ABX = A\lambda X = \lambda AX$. Here it is assumed that $BX = \lambda X$.

Problem 4 Suppose A is an $n \times n$ matrix and it satisfies $A^m = A$ for some m a positive integer larger than 1. Show that if λ is an eigenvalue of A then $|\lambda|$ equals either 0 or 1.

Click the arrow to see answer. Let X be the eigenvector. Then $A^m X = \lambda^m X$, $A^m X = AX = \lambda X$ and so

$$\lambda^m = \lambda$$

Hence if $\lambda \neq 0$, then

$$\lambda^{m-1} = 1$$

and so $|\lambda| = 1$.

Problem 5 Show that if $AX = \lambda X$ and $AY = \lambda Y$, then whenever k, p are scalars,

$$A(kX + pY) = \lambda(kX + pY)$$

Does this imply that $kX + pY$ is an eigenvector? Explain.

Click the arrow to see answer. The formula follows from properties of matrix multiplications. However, this vector might not be an eigenvector because it might equal 0 and eigenvectors cannot equal 0.

Problem 6 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 6 & 76 & 16 \\ -2 & -21 & -4 \\ 2 & 64 & 17 \end{bmatrix}$$

One eigenvalue is -2 .

Click the arrow to see answer. The characteristic polynomial of this matrix is $-\lambda^3 + 2\lambda^2 + 5\lambda - 6$. Knowing one of the eigenvalues gives us one factor, $(\lambda + 2)$. Use long division of polynomials to finish factoring $-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = -(\lambda + 2)(\lambda - 1)(\lambda - 3)$. Now we have the following eigenvalues: $\lambda_1 = -2$, $\lambda_2 = 1$, and $\lambda_3 = 3$.

The corresponding eigenvectors are: $\mathbf{v}_1 = \begin{bmatrix} 7 \\ -2 \\ 6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$.

Problem 7 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & 5 & 2 \\ -8 & -11 & -4 \\ 10 & 11 & 3 \end{bmatrix}$$

One eigenvalue is -3 .

Click the arrow to see answer. Characteristic polynomial: $-\lambda^3 - 5\lambda^2 - 7\lambda - 3 = -(\lambda + 3)(\lambda + 1)^2$.

$$\lambda_1 = -3, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$\lambda_2 = -1, \quad \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Problem 8 Is it possible for a nonzero matrix to have only 0 as an eigenvalue?

Click the arrow to see answer. Yes. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ works.

Problem 9 Let T be the linear transformation which reflects vectors about the x axis. Find a matrix for T and then find its eigenvalues and eigenvectors.

Click the arrow to see answer.

The matrix of T is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

The eigenvalues and eigenvectors are:

$$\lambda_1 = -1, \quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Problem 10 Let T be the linear transformation which reflects all vectors in $\mathbb{R}^3$ through the xy plane. Find a matrix for T and then obtain its eigenvalues and eigenvectors.

Click the arrow to see answer. The matrix of T is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ The eigenvalues are $\lambda_1 = -1, \lambda_2 = 1$. Bases for the corresponding eigenspaces are:

$$S_{\lambda_1} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, S_{\lambda_2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Problem 11 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 5 & -18 & -32 \\ 0 & 5 & 4 \\ 2 & -5 & -11 \end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible.

Click the arrow to see answer. The eigenvalues are $-1, -1, 1$. The eigenvectors corresponding to the eigenvalues are:

$$\left\{ \begin{bmatrix} 10 \\ -2 \\ 3 \end{bmatrix} \right\} \leftrightarrow -1, \left\{ \begin{bmatrix} 7 \\ -2 \\ 2 \end{bmatrix} \right\} \leftrightarrow 1$$

Therefore this matrix is not diagonalizable.

Problem 12 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix}$$

One eigenvalue is 3. Diagonalize if possible.

Click the arrow to see answer. The eigenvectors and eigenvalues are:

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} \leftrightarrow 1, \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\} \leftrightarrow 1, \left\{ \begin{bmatrix} 7 \\ -2 \\ 2 \end{bmatrix} \right\} \leftrightarrow 3$$

The matrix P needed to diagonalize the above matrix is

$$\begin{bmatrix} 2 & -2 & 7 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

and the diagonal matrix D is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Problem 13 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix}$$

One eigenvalue is -3 . Diagonalize if possible.

Click the arrow to see answer. The eigenvectors and eigenvalues are:

$$\left\{ \begin{bmatrix} -6 \\ -1 \\ -2 \end{bmatrix} \right\} \leftrightarrow 6, \left\{ \begin{bmatrix} -5 \\ -2 \\ 2 \end{bmatrix} \right\} \leftrightarrow -3, \left\{ \begin{bmatrix} -8 \\ -2 \\ 3 \end{bmatrix} \right\} \leftrightarrow -2$$

The matrix P needed to diagonalize the above matrix is

$$\begin{bmatrix} -6 & -5 & -8 \\ -1 & -2 & -2 \\ 2 & 2 & 3 \end{bmatrix}$$

and the diagonal matrix D is

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Problem 14 If $A \sim B$ and A has any of the following properties, show that B has the same property.

1. A is Idempotent, that is $A^2 = A$.

2. A is Nilpotent, that is $A^k = 0$ for some $k \geq 1$.

Click the arrow to see answer. If $B = P^{-1}AP$ and $A^k = 0$, then $B^k = (P^{-1}AP)^k = P^{-1}A^kP = P^{-1}0P = 0$.

3. A is Invertible.

Bibliography

These problems come from Chapter 7 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 359–401.