
1 Solved Problems

Solved Problems for Chapter 6

Problem 1 Show that the function T_u defined by $T_u(\mathbf{v}) = \mathbf{v} - \text{proj}_u(\mathbf{v})$ is also a linear transformation.

Click on the arrow to see answer.

$$\begin{aligned} T_u(a\mathbf{v} + b\mathbf{w}) &= a\mathbf{v} + b\mathbf{w} - \frac{(a\mathbf{v} + b\mathbf{w}) \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u} = \\ &= a\mathbf{v} - a \frac{(\mathbf{v} \cdot \mathbf{u})}{\|\mathbf{u}\|^2} \mathbf{u} + b\mathbf{w} - b \frac{(\mathbf{w} \cdot \mathbf{u})}{\|\mathbf{u}\|^2} \mathbf{u} = aT_u(\mathbf{v}) + bT_u(\mathbf{w}) \end{aligned}$$

Problem 2 Let $\mathbf{u}$ be a fixed vector. The function T_u defined by $T_u\mathbf{v} = \mathbf{u} + \mathbf{v}$ has the effect of translating all vectors by adding $\mathbf{u} \neq \mathbf{0}$. Show that T_u is not a linear transformation.

Click the arrow to see answer.

Linear transformations take $\mathbf{0}$ to $\mathbf{0}$ which T does not. Also $T_a(\mathbf{u} + \mathbf{v}) \neq T_a\mathbf{u} + T_a\mathbf{v}$.

Problem 3 Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/3$.

Click on the arrow to see answer.

$$\begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$$

Problem 4 Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $2\pi/3$ and then reflects across the x axis.

Click on the arrow to see answer.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$$

Problem 5 Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/6$ and then reflects across the x axis followed by a reflection across the y axis.

Click on the arrow to see answer.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

Problem 6 Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and let S be a linear transformation induced by $B = \begin{bmatrix} 0 & -2 \\ 4 & 2 \end{bmatrix}$. Find matrix of $S \circ T$ and find $(S \circ T)(\mathbf{x})$ for $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Click on the arrow to see answer. The matrix of $S \circ T$ is given by BA .

$$\begin{bmatrix} 0 & -2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 10 & 8 \end{bmatrix}$$

Now, $(S \circ T)(\mathbf{x}) = (BA)\mathbf{x}$.

$$\begin{bmatrix} 2 & -4 \\ 10 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

Problem 7 Let T be a linear transformation and suppose $T\left(\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Suppose S is a linear transformation induced by the matrix $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Find $(S \circ T)(\mathbf{x})$ for $\mathbf{x} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

Click on the arrow to see answer. To find $(S \circ T)(\mathbf{x})$ we compute $S(T(\mathbf{x}))$.

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 \\ -11 \end{bmatrix}$$

Problem 8 Let T be a linear transformation given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find a basis for $\ker(T)$ and a basis for $\text{im}(T)$. Find the dimension of the kernel and the image of T .

Click on the arrow to see answer. A basis for $\ker(T)$ is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and a basis for $\text{im}(T)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. There are many other possibilities for the specific bases. $\dim(\ker(T)) = 1$ and $\dim(\text{im}(T)) = 1$.

Problem 9 Let T be a linear transformation given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find a basis for $\ker(T)$ and a basis for $\text{im}(T)$.

Click on the arrow to see answer.

In this case $\ker(T) = \{0\}$ and $\text{im}(T) = \mathbb{R}^2$ (pick any basis of $\mathbb{R}^2$).

Problem 10 Let T be a linear transformation given by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

What is $\dim(\ker(T))$?

Click on the arrow to see answer. We can easily see that $\dim(\text{im}(T)) = 1$, and thus $\dim(\ker(T)) = 3 - \dim(\text{im}(T)) = 3 - 1 = 2$.

Problem 11 Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a linear transformation such that $T \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ -6 \\ 1 \end{bmatrix}$, and $T \left(\begin{bmatrix} 4 \\ 1 \end{bmatrix} \right) =$

$\begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$. Find the image of $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$ under T .

Click the arrow to see answer.

First, observe that $\begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Therefore,

$$T \left(\begin{bmatrix} -2 \\ -2 \end{bmatrix} \right) = T \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} \right) - T \left(\begin{bmatrix} 4 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ -6 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -11 \\ 1 \end{bmatrix}$$

Problem 12 Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation that maps $\mathbf{i}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{j}$ to $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find vector $\mathbf{v}$

such that $T(\mathbf{v}) = \begin{bmatrix} 20 \\ -30 \end{bmatrix}$.

Click the arrow to see answer.

Observe that T is induced by the matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$. We are looking for $\mathbf{v}$ such that $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 20 \\ -30 \end{bmatrix}$.

Multiplying both sides by A^{-1} gives us

$$\mathbf{v} = \begin{bmatrix} 0.6 & 0.4 \\ -0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 20 \\ -30 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \end{bmatrix}$$

Bibliography

Some of the problems come from Chapter 5 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 272–315.