
1 Solved Problems

Solved Problems for Chapter 4

Problem 1 Using only the properties given in Theorem ?? and Theorem ??, show that the additive inverse of A , $-A$, is unique.

Click on the arrow to see answer. Suppose B is also an additive inverse of A . Then

$$-A = -A + (A + B) = (-A + A) + B = 0 + B = B$$

Problem 2 Using only the properties given in Theorem ?? and Theorem ??, show that the $n \times m$ zero matrix, O , is unique.

Click on the arrow to see answer. Suppose O' is also an $n \times m$ additive identity. Then $O' = O' + O = O$.

Problem 3 Using only the properties given in Theorem ?? and Theorem ??, show that $0A = O$.

Click on the arrow to see answer. $0A = (0 + 0)A = 0A + 0A$. Now add $-(0A)$ to both sides. Then $O = 0A$.

Problem 4 Using only the properties given in Theorem ?? and Theorem ??, as well as previous problems, show $(-1)A = -A$.

Click on the arrow to see answer. $A + (-1)A = (1 + (-1))A = 0A = O$. Therefore, from the uniqueness of the additive inverse proved in the above Problem ??, it follows that $-A = (-1)A$.

Problem 5 Consider the matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ -3 & 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$,
 $D = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$, $E = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Find the following if possible. If it is not possible explain why.

1. $-3A$

Click on the arrow to see answer. $\begin{bmatrix} -3 & -6 & -9 \\ -6 & -3 & -21 \end{bmatrix}$

2. $3B - A$

Click on the arrow to see answer. $\begin{bmatrix} 8 & -5 & 3 \\ -11 & 5 & -4 \end{bmatrix}$

3. AC

Click on the arrow to see answer. Not possible

4. CB

Click on the arrow to see answer. $\begin{bmatrix} -3 & 3 & 4 \\ 6 & -1 & 7 \end{bmatrix}$

5. AE

Click on the arrow to see answer. Not possible

6. EA

Click on the arrow to see answer. Not possible

Problem 6 Let $A = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$. Find all 2×2 matrices, B such that $AB = O$.

Click on the arrow to see answer.

$$\begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} -x-z & -w-y \\ 3x+3z & 3w+3y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution is: $w = -y, x = -z$ so the matrices are of the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$.

Problem 7 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix}$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?

Click on the arrow to see answer.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 7 & 2k+2 \\ 15 & 4k+6 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3k+3 & 4k+6 \end{bmatrix}$$

Thus you must have $\begin{matrix} 3k+3=15 \\ 2k+2=10 \end{matrix}$. Therefore $k = 4$.

Problem 8 Find 2×2 matrices A and B such that $A \neq O$ and $B \neq O$ but $AB = O$.

Click on the arrow to see answer. Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 9 Suppose $AB = AC$ and A is an invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

Click on the arrow to see answer. Yes $B = C$. Multiply $AB = AC$ on the left by A^{-1} .

Problem 10 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Click on the arrow to see answer. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 4 & -5 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$

Problem 11 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 10 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Click on the arrow to see answer. The reduced row echelon form is $\begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$. There is no inverse.

Problem 12 Show that if A^{-1} exists for an $n \times n$ matrix, then it is unique. That is, if $BA = I$ and $AB = I$, then $B = A^{-1}$.

Click on the arrow to see answer. $A^{-1} = A^{-1}I = A^{-1}(AB) = (A^{-1}A)B = IB = B$.

Problem 13 Show that if A is an invertible $n \times n$ matrix, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

Click the arrow to see answer.

You need to show that $(A^{-1})^T$ acts like the inverse of A^T because from uniqueness in the above problem, this will imply it is the inverse. From properties of the transpose,

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

Hence $(A^{-1})^T = (A^T)^{-1}$ and this last matrix exists.

Problem 14 Show $(AB)^{-1} = B^{-1}A^{-1}$ by verifying that

$$AB(B^{-1}A^{-1}) = I$$

and

$$B^{-1}A^{-1}(AB) = I$$

Click on the arrow to see answer. $(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I$ $B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

Problem 15 Find an LU factorization of the coefficient matrix and use it to solve the system of equations.

$$x + 2y = 5$$

$$2x + 3y = 6$$

Click on the arrow to see answer. An LU factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

First solve

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

which gives $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$. Then solve

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

which says that $y = 4$ and $x = -3$.

Problem 16 Find an LU factorization of the coefficient matrix and use it to solve the system of equations.

$$x + 2y + z = 1$$

$$y + 3z = 2$$

$$2x + 3y = 6$$

Click on the arrow to see answer. An LU factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

First solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

which yields $u = 1, v = 2, w = 6$. Next solve

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

This yields $z = 6, y = -16, x = 27$.

Bibliography

These problems come from the end of Chapter 2 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 90–98, 104–106.