
1 Solved Problems

Solved Problems for Chapter 9

Problem 1 Find an orthonormal basis for the span of the following set of vectors.

$$\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$$

Click the arrow to see answer. We will use Gram-Schmidt orthogonalization algorithm.

$$\mathbf{f}_1 = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$$

$$\mathbf{f}_2 = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} - \frac{25}{25} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

$$\mathbf{f}_3 = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix} - \frac{-25}{25} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} - \frac{25}{25} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Normalizing each vector we get:

$$\begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Problem 2 Using the Gram Schmidt process find an orthonormal basis for the following span:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Click the arrow to see the answer.

$$\begin{bmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}, \begin{bmatrix} \frac{3}{10}\sqrt{2} \\ -\frac{2}{5}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} \frac{7}{15}\sqrt{3} \\ -\frac{1}{15}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} \end{bmatrix}$$

Problem 3 Write $\mathbf{x}$ as the sum of a vector in W and a vector in $W^\perp$ if

$$\mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, \quad W = \text{span} \left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Click the arrow to see the answer. Observe that the two vectors that span W are orthogonal. We will refer to these vectors as $\mathbf{f}_1$ and $\mathbf{f}_2$.

$$\mathbf{w} = \text{proj}_W(\mathbf{x}) = \text{proj}_{\mathbf{f}_1}(\mathbf{x}) + \text{proj}_{\mathbf{f}_2}(\mathbf{x}) = \frac{12}{14} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \frac{11}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} -59 \\ 41 \\ 131 \end{bmatrix}$$

$$\mathbf{w}^\perp = \mathbf{x} - \mathbf{w} = \frac{1}{21} \begin{bmatrix} 80 \\ 64 \\ 16 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{w} + \mathbf{w}^\perp$$

Problem 4 Write $\mathbf{x}$ as the sum of a vector in W and a vector in $W^\perp$ if

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 9 \end{bmatrix}, \quad W = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

Click the arrow to see the answer. Observe that the three vectors that span W form an orthogonal set. We will refer to these vectors as $\mathbf{f}_1$, $\mathbf{f}_2$ and $\mathbf{f}_3$.

$$\mathbf{w} = \text{proj}_W(\mathbf{x}) = \text{proj}_{\mathbf{f}_1}(\mathbf{x}) + \text{proj}_{\mathbf{f}_2}(\mathbf{x}) + \text{proj}_{\mathbf{f}_3}(\mathbf{x}) =$$

$$= \frac{17}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{5}{3} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{8}{6} \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 9 \\ 5 \\ 16 \\ 26 \end{bmatrix}$$

$$\mathbf{w}^\perp = \mathbf{x} - \mathbf{w} = \frac{1}{3} \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{w} + \mathbf{w}^\perp$$

Bibliography

Some of the problems come from the end of Chapter 7 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 433–438.