
1 Solved Problems

Solved Problems for Chapter 5

Problem 1 Let $M = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : |u_1| \leq 4 \right\}$. Is M a subspace? Explain.

Click the arrow to see answer. No. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is in M but $10 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is not in M .

Problem 2 Let $M = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_i \geq 0 \text{ for each } i = 1, 2, 3, 4 \right\}$. Is M a subspace? Explain.

Click the arrow to see answer. This is not a subspace. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is in M . However, $(-1) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not in M .

Problem 3 Let $\mathbf{w} \in \mathbb{R}^4$ and let $M = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : \mathbf{w} \cdot \mathbf{u} = 0 \right\}$. Is M a subspace? Explain.

Click the arrow to see answer. Yes, this is a subspace. First, M is not empty because the zero vector is in M . You can verify closure under vector addition and scalar multiplication using the properties of the dot product as follows. If $\mathbf{u}$ is in M , then $\mathbf{w} \cdot \mathbf{u} = 0$ and $\mathbf{w} \cdot k\mathbf{u} = 0$ for all constants k . Finally, if $\mathbf{u}_1$ and $\mathbf{u}_2$ are in M , then $\mathbf{w} \cdot (\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{w} \cdot \mathbf{u}_1 + \mathbf{w} \cdot \mathbf{u}_2 = 0$.

Problem 4 Let $M = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_3 = u_1 = 0 \right\}$. Is M a subspace? Explain.

Click the arrow to see answer. This is a subspace. M is not empty because the zero vector is in it. M is closed under vector addition and scalar multiplication.

Problem 5 If you have 5 vectors in $\mathbb{R}^5$ and the vectors are linearly independent, can it always be concluded they span $\mathbb{R}^5$? Explain.

Click the arrow to see answer. Yes. If not, there would exist a vector not in the span. But then you could add in this vector and obtain a linearly independent set of vectors with more vectors than a basis.

Problem 6 If you have 6 vectors in $\mathbb{R}^5$, is it possible they are linearly independent? Explain.

Click the arrow to see answer. A matrix A with five rows and six columns will contain a non-pivot column, giving rise to a non-trivial solution (infinitely many of them) to the equation $A\mathbf{x} = \mathbf{0}$. This shows that the columns of A are linearly dependent.

Problem 7 Suppose V, W are subspaces of $\mathbb{R}^n$. Let $V \cap W$ be all vectors which are in both V and W . Show that $V \cap W$ is a subspace also.

Click the arrow to see answer. If $\mathbf{x}, \mathbf{y} \in V \cap W$, then for scalars α, β , the linear combination $\alpha\mathbf{x} + \beta\mathbf{y}$ must be in both V and W since they are both subspaces.

Problem 8 Find the rank of the following matrix, and find a basis for the column spaces.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 3 & 1 & 10 & 0 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

Click the arrow to see answer.

$$\text{rref} \left(\begin{bmatrix} 1 & 0 & 3 & 0 \\ 3 & 1 & 10 & 0 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 2 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the matrix is 3. We can use columns 1, 2, and 4 as a basis for the column space.

Problem 9 Find a basis for $\text{null}(A)$.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

Click the arrow to see answer.

$$\text{rref} \left(\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the null space is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Problem 10 Find a basis for $\text{null}(A)$

$$A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 2 & 0 & 1 & 2 \\ 6 & 4 & -5 & -6 \\ 0 & 2 & -4 & -6 \end{bmatrix}$$

Click the arrow to see answer.

$$\text{rref} \left(\begin{bmatrix} 2 & -1 & 3 & 5 \\ 2 & 0 & 1 & 2 \\ 6 & 4 & -5 & -6 \\ 0 & 2 & -4 & -6 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for the null space is $\left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}.$

Problem 11 Suppose matrix A has linearly independent columns. What is the null space of A ?

Click the arrow to see answer.

The only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial one. Therefore, the only vector in $\text{null}(A)$ is the zero vector.

Problem 12 Let $V = \text{span} \left(\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix} \right)$. Find a basis for V . What is the dimension of V ?

Click the arrow to see answer.

Form a matrix A using the given vectors as columns. $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. The leading 1's are in the first two columns. We can use the first two of the given vectors to form a basis of V . Therefore $\dim(V) = 2$.

Bibliography

Some of the problems come from the end of Chapter 4 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY) Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 227–232.