

---

# 1 Solved Problems

## Solved Problems for Chapter 2

**Problem 1** You have a system of  $k$  equations in two variables,  $k \geq 2$ . Explain the geometric significance of

1. No solution.  
*The  $k$  lines do not have a point common to all of them.*
2. A unique solution.  
*All  $k$  lines intersect at a single point.*
3. An infinite number of solutions.  
*The  $k$  lines coincide.*

*Click the arrow to see answer.*

**Problem 2** Consider the following augmented matrix in which  $*$  denotes an arbitrary number and  $\blacksquare$  denotes a nonzero number. Determine whether the given augmented matrix corresponds to a consistent system. If consistent, is the solution unique?

$$\left[ \begin{array}{cccc|c} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & 0 \\ 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & \blacksquare \end{array} \right]$$

*Click the arrow to see answer. The third equation implies that  $x_5 = 0$ . The fourth equation implies that  $x_5 \neq 0$ . We conclude that the system is inconsistent.*

**Problem 3** Suppose a system of equations has fewer equations than variables. Will such a system necessarily be consistent? If so, explain why and if not, give an example which is not consistent.

*Click the arrow to see answer. No. Consider  $x + y + z = 2$  and  $x + y + z = 1$ .*

**Problem 4** If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, explain why not.

*Click the arrow to see answer. These can have a solution. For example,  $x + y = 1, 2x + 2y = 2, 3x + 3y = 3$  even has an infinite set of solutions.*

**Problem 5** Find  $h$  such that

$$\left[ \begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array} \right]$$

*is the augmented matrix of an inconsistent system.*

---


$$\left[ \begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array} \right] \rightsquigarrow \left[ \begin{array}{cc|c} 3 & (3/2)h & 6 \\ 3 & 6 & 7 \end{array} \right]. \text{ The system is inconsistent for } h = 4.$$

**Problem 6** Find  $h$  such that

$$\left[ \begin{array}{cc|c} 1 & h & 3 \\ 2 & 4 & 6 \end{array} \right]$$

is the augmented matrix of a consistent system.

Click the arrow to see answer. The reduced row-echelon form will never have a row of the form  $[0 \ 0 \ | \ 1]$ . The system is consistent for all  $h$ .

**Problem 7** Choose  $h$  and  $k$  such that the augmented matrix shown has each of the following:

1. one solution
2. no solution
3. infinitely many solutions

$$\left[ \begin{array}{cc|c} 1 & 2 & 2 \\ 2 & h & k \end{array} \right]$$

Click the arrow to see answer. If  $h \neq 4$ , then there is exactly one solution. If  $h = 4$  and  $k \neq 4$ , then there are no solutions. If  $h = 4$  and  $k = 4$ , then there are infinitely many solutions.

**Problem 8** Determine if the system is consistent. If so, is the solution unique?

$$\begin{array}{rrcrcl} x & + & 2y & + & z & - & w & = & 2 \\ x & - & y & + & z & + & w & = & 1 \\ 2x & + & y & - & z & & & = & 1 \\ 4x & + & 2y & + & z & & & = & 5 \end{array}$$

Click the arrow to see answer. There is no solution because the rref of the augmented matrix contains a row of the form  $[0 \ 0 \ 0 \ 0 \ | \ 1]$

$$\text{rref} \left( \left[ \begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 0 & 1 \\ 4 & 2 & 1 & 0 & 5 \end{array} \right] \right) = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

**Problem 9** Determine if the system is consistent. If so, is the solution unique?

$$\begin{array}{rrcrcl} x & + & 2y & + & z & - & w & = & 2 \\ x & - & y & + & z & + & w & = & 0 \\ 2x & + & y & - & z & & & = & 1 \\ 4x & + & 2y & + & z & & & = & 3 \end{array}$$

Click the arrow to see answer.

$$\text{rref} \left( \left[ \begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 1 \\ 4 & 2 & 1 & 0 & 3 \end{array} \right] \right) = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & -2/3 & 2/3 \\ 0 & 0 & 1 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

There are infinitely many solutions:  $w = t$ ,  $z = \frac{1}{3}$ ,  $y = \frac{2}{3} + \frac{2}{3}t$ ,  $x = \frac{1}{3} - \frac{1}{3}t$ .

**Problem 10** Find the solution of the system whose augmented matrix is

$$\left[ \begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 \end{array} \right]$$

Click the arrow to see answer.

$$\text{rref} \left( \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 6/5 \\ 0 & 1 & 0 & 2/5 \\ 0 & 0 & 1 & -1/10 \end{array} \right]$$

$$x = \frac{6}{5}, y = \frac{2}{5}, z = -\frac{1}{10}$$

**Problem 11** Solve the system whose augmented matrix is

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 4 & 2 \end{array} \right]$$

Click the arrow to see answer. The reduced row echelon form is  $\left[ \begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & -4 & -1 \end{array} \right]$  and so the solution is  $z = t$ ,  $y = 4t$ ,  $x = 2 - 4t$ .

**Problem 12** Solve the system if the rref of its augmented matrix is

$$\left[ \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 9 & 3 \\ 0 & 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & -7 & -1 \\ 0 & 0 & 0 & 1 & 6 & 1 \end{array} \right]$$

Click the arrow to see answer.  $x_5 = t$ ,  $x_4 = 1 - 6t$ ,  $x_3 = -1 + 7t$ ,  $x_2 = 4t$ ,  $x_1 = 3 - 9t$ .

**Problem 13** Solve the system if the rref of its augmented matrix is

$$\left[ \begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1/2 & 5/2 \\ 0 & 1 & 0 & 0 & 1/2 & 3/2 \\ 0 & 0 & 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

---

Click the arrow to see answer. The free variables are  $x_5 = t$ ,  $x_3 = s$ . The other variables are given by  $x_4 = -\frac{1}{2} - \frac{3}{2}t$ ,  $x_2 = \frac{3}{2} - \frac{1}{2}t$ ,  $x_1 = \frac{5}{2} + \frac{1}{2}t - 2s$ .

**Problem 14** Suppose a system of equations has fewer equations than variables and you have found a solution to this system of equations. Is it possible that your solution is the only one? Explain.

Click the arrow to see answer. No. The rank of the coefficient matrix in this case is smaller than the number of columns (variables). So, there has to be a free variable. The parameter ( $t$  is a typical choice) assigned to the free variable will guarantee infinitely many solutions.

**Problem 15** Suppose  $A$  is an  $m \times n$  matrix. Explain why the rank of  $A$  is always no larger than  $\min(m, n)$ .

Click the arrow to see answer. It is because you cannot have more leading 1's than columns and you cannot have more leading 1's than rows.

**Problem 16** Find the rank of the following matrix.

$$\begin{bmatrix} 4 & 4 & 20 & -1 & 17 \\ 1 & 1 & 5 & 0 & 5 \\ 1 & 1 & 5 & -1 & 2 \\ 3 & 3 & 15 & -3 & 6 \end{bmatrix}$$

Click the arrow to see answer.

$$\text{rref} \left( \begin{bmatrix} 4 & 4 & 20 & -1 & 17 \\ 1 & 1 & 5 & 0 & 5 \\ 1 & 1 & 5 & -1 & 2 \\ 3 & 3 & 15 & -3 & 6 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 & 5 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank is the number of leading 1's. The rank of the given matrix is 2.

## Bibliography

These problems come from the end of Chapter 1 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, pp. 42–49.