
1 Homogeneous Linear Systems

Definition 1. A system of linear equations is called *homogeneous* if the system can be written in the form

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & 0 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & 0 \end{array}$$

A homogeneous linear system is always consistent because $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is a solution. This solution is called the *trivial solution*. Geometrically, a homogeneous system can be interpreted as a collection of lines or planes (or hyperplanes) passing through the origin. Thus, they will always have the origin in common, but may have other points in common as well.

If A is the coefficient matrix for a homogeneous system, then the system can be written as a matrix equation $A\mathbf{x} = \mathbf{0}$. The augmented matrix that represents the system looks like this

$$\left[A \mid 0 \right]$$

As we perform elementary row operations, the entries to the right of the vertical bar remain 0. It is customary to omit writing them down and apply elementary row operations to the coefficient matrix only.

Example 1. Solve the given homogeneous system and interpret your solution geometrically.

$$\begin{array}{rrcrcl} 4x & + & 5y & - & z & = & 0 \\ x & - & 4y & - & 2z & = & 0 \\ 3x & - & 6y & - & 4z & = & 0 \end{array}$$

Explanation. We start by rewriting this system as a matrix equation

$$\begin{bmatrix} 4 & 5 & -1 \\ 1 & -4 & -2 \\ 3 & -6 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

We will proceed to find the reduced row-echelon form of the matrix as usual, but will omit writing the zeros to the right of the vertical bar.

$$\begin{bmatrix} 4 & 5 & -1 \\ 1 & -4 & -2 \\ 3 & -6 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

x and y are leading variables, and z is a free variable. We let $z = t$ and solve for x and y .

$$\begin{array}{l} x = \frac{2}{3}t \\ y = -\frac{1}{3}t \\ z = t \end{array}$$

Each of the equations in the original system represents a plane through the origin in $\mathbb{R}^3$. The system has infinitely many solutions. Geometrically, we can interpret these solutions as points lying on the line shared by the three planes. The above solution is a parametric representation of this line. Use the GeoGebra demo below to take a better look at the system. (RIGHT-CLICK and DRAG to rotate the image.)

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

General and Particular Solutions

Definition 2. Given any linear system $A\mathbf{x} = \mathbf{b}$, the system $A\mathbf{x} = \mathbf{0}$ is called the *associated homogeneous system*.

It turns out that there is a relationship between solutions of $A\mathbf{x} = \mathbf{b}$ and solutions of the associated homogeneous system.

Exploration 1 Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -7 & -1 \\ -1 & 4 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -2 \\ 7 \\ -4 \end{bmatrix}$$

Consider the matrix equation $A\mathbf{x} = \mathbf{b}$. Row reduction produces the following.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & -2 \\ 3 & -7 & -1 & 7 \\ -1 & 4 & 2 & -4 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

We conclude that $\mathbf{x} = \begin{bmatrix} -2t \\ -1 - t \\ t \end{bmatrix}$.

Let's take a more careful look at $\mathbf{x}$.

$$\mathbf{x} = \begin{bmatrix} -2t \\ -1 - t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ \dots \end{bmatrix} + \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix} t$$

We now see that the solution vector $\mathbf{x}$ is made up of two distinct parts:

- one specific vector $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$
- infinitely many scalar multiples of $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

The vector $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ is an example of a particular solution. This particular “particular solution” corresponds to $t = 0$. We can find any number of particular solutions by letting t assume different values. For example, the particular solution that corresponds to $t = 1$ is $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$. Let $\mathbf{x}_p$ be any particular solution of $A\mathbf{x} = \mathbf{b}$. It turns out that all vectors of the form

$$\mathbf{x} = \mathbf{x}_p + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} t$$

are solutions of $A\mathbf{x} = \mathbf{b}$. We can verify this as follows

$$A\mathbf{x} = A \left(\mathbf{x}_p + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} t \right) = A\mathbf{x}_p + \begin{bmatrix} 1 & 2 & 4 \\ 3 & -7 & -1 \\ -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} t = A\mathbf{x}_p + \mathbf{0} = \mathbf{b}$$

This shows that the specific vector $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ is not very special, as any solution of $A\mathbf{x} = \mathbf{b}$ can be used in its place.

The vector $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, however, is special. Note that

$$A \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -7 & -1 \\ -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \mathbf{0}$$

So $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ and all of its scalar multiples are solutions to the associated homogeneous system.

Observation 1. In Exploration ?? we found that the general solution of the equation $A\mathbf{x} = \mathbf{b}$ has the form:

$$\mathbf{x} = (\text{Any Particular Solution of } A\mathbf{x} = \mathbf{b}) + (\text{General Solution of } A\mathbf{x} = \mathbf{0})$$

It turns out that the general solution of any linear system can be written in this format. Theorem ?? formalizes this result.

Theorem 1. Suppose $\mathbf{x}_p$ is a particular solution of $A\mathbf{x} = \mathbf{b}$.

1. If $\mathbf{x}_h$ is a solution of the associated homogeneous system, then $\mathbf{x}_p + \mathbf{x}_h$ is a solution of $A\mathbf{x} = \mathbf{b}$.
2. If $\mathbf{x}_1$ is a solution of $A\mathbf{x} = \mathbf{b}$, then there exists a solution of the associated homogeneous system, $\mathbf{x}_h$, such that $\mathbf{x}_1 = \mathbf{x}_p + \mathbf{x}_h$.

We will prove part ???. The proof of part ??? is left to the reader.

Proof of ??? Let $\mathbf{x}_h = \mathbf{x}_1 - \mathbf{x}_p$, then

$$A\mathbf{x}_h = A(\mathbf{x}_1 - \mathbf{x}_p) = A\mathbf{x}_1 - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

and

$$\mathbf{x}_1 = \mathbf{x}_p + \mathbf{x}_h$$

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Example 2. Let

$$A = \begin{bmatrix} 2 & -4 & -2 \\ 1 & -2 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

If possible, find a solution of $A\mathbf{x} = \mathbf{b}$ and express it as a sum of a particular solution and the general solution of the associated homogeneous system. ($\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$)

Explanation. First, we obtain the reduced row-echelon form

$$\left[\begin{array}{ccc|c} 2 & -4 & -2 & 8 \\ 1 & -2 & -1 & 4 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

So

$$\mathbf{x} = \begin{bmatrix} 4 + 2s + t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t$$

In this case

$$\mathbf{x}_p = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_h = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t$$

Practice Problems

Problems ???-???

For each matrix A and vector $\mathbf{b}$ below, find a solution to $A\mathbf{x} = \mathbf{b}$ and express your solution as a sum of a particular solution and a general solution to the associated homogeneous system.

Problem 2

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & 4 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

Problem 3

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 1 \\ 4 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 10 \\ 2 \\ 12 \end{bmatrix}$$

Problem 4 *Prove that a consistent system has infinitely many solutions if and only if the associated homogeneous system has infinitely many solutions.*

Problems ??-??

If possible, construct an example of each of the following. If not possible, explain why.

Problem 5 *An inconsistent system with an associated homogeneous system that has infinitely many solutions.*

Problem 6 *An inconsistent system with an associated homogeneous system that has a unique (trivial) solution.*

Problem 7 *Prove that a linear combination of any number of solutions of a homogeneous equation is a solution of the same equation.*

Problem 8 *Prove Part ?? of Theorem ??.*