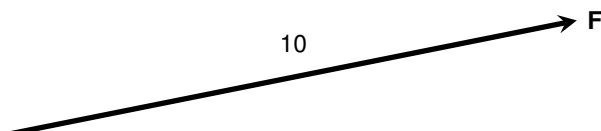


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# 1 Length of a Vector

Vector quantities, such as velocity and force, have magnitude and direction. The magnitude of a vector quantity is the length of the vector. For example, if a force of 10 Newtons is applied to an object, we would represent the force by a 10-unit-long vector.



The magnitude of a vector is denoted by double absolute value brackets. In the case of force **F**, we write

$$\|\mathbf{F}\| = 10$$

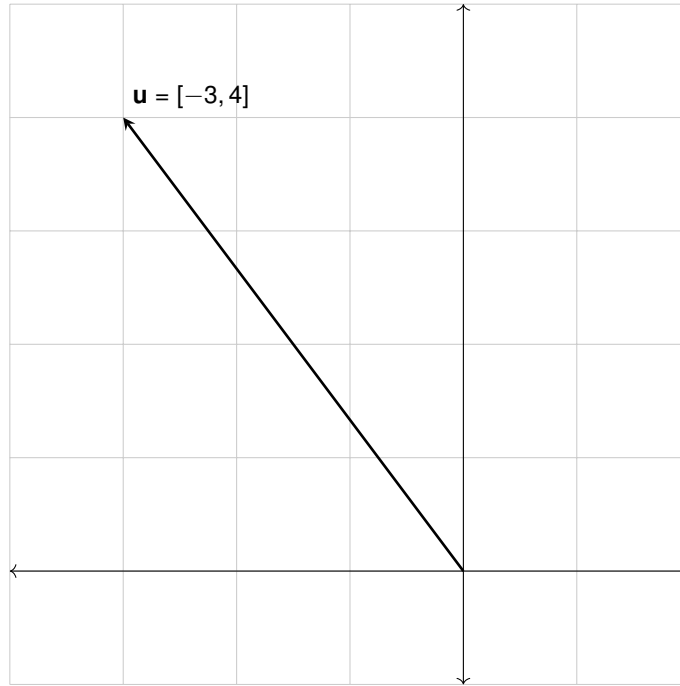
To find the length of a vector, we need to find the distance between the tail of the vector and its head. Recall that in  $\mathbb{R}^2$ , the distance between  $A(a_1, a_2)$  and  $B(b_1, b_2)$  is given by

$$d_{AB} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

A vector  $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  has the length of the vector in standard position with its head at  $(v_1, v_2)$  and tail at  $(0, 0)$ . We find the length of  $\mathbf{v}$  using the distance formula

$$\|\mathbf{v}\| = \sqrt{(v_1 - 0)^2 + (v_2 - 0)^2} = \sqrt{v_1^2 + v_2^2} \quad (1)$$

**Example 1.** Find the magnitude of  $\mathbf{u} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ .



**Explanation.**

$$\|\mathbf{u}\| = \sqrt{(-3)^2 + (4)^2} = 5$$

The distance formula for points in  $\mathbb{R}^3$  is analogous to the distance formula in  $\mathbb{R}^2$ . Given two points  $A(a_1, a_2, a_3)$  and  $B(b_1, b_2, b_3)$ , the distance between them is given by

$$d_{AB} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

To find the length of vector  $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ , we find the distance between  $(v_1, v_2, v_3)$  and  $(0, 0, 0)$ .

$$\|\mathbf{v}\| = \sqrt{(v_1 - 0)^2 + (v_2 - 0)^2 + (v_3 - 0)^2} = \sqrt{v_1^2 + v_2^2 + v_3^2} \quad (2)$$

**Example 2.** The magnitude of  $\mathbf{v} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$  is given by

$$\|\mathbf{v}\| = \sqrt{(6)^2 + (-2)^2 + (3)^2} = 7$$

Distance formulas for  $\mathbb{R}^2$  and  $\mathbb{R}^3$  motivate the following definition of distance between two points in  $\mathbb{R}^n$ .

**Definition 1.** Let  $A(a_1, a_2, \dots, a_n)$  and  $B(b_1, b_2, \dots, b_n)$  be points in  $\mathbb{R}^n$ . The *distance* between  $A$  and  $B$  is defined to be

$$d_{AB} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \dots + (a_n - b_n)^2}$$

The following definition follows directly from the distance formula for  $\mathbb{R}^n$  in the same way that expressions (??) and (??) followed from distance formulas in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ .

**Definition 2.** Let  $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$  be a vector in  $\mathbb{R}^n$ , then the *length*, or the *magnitude*, of  $\mathbf{v}$  is given by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} \quad (3)$$

**Example 3.** The magnitude of  $\mathbf{w} = \begin{bmatrix} 5 \\ -1 \\ -2 \\ 4 \end{bmatrix}$  is given by

$$\|\mathbf{w}\| = \sqrt{5^2 + (-1)^2 + (-2)^2 + 4^2} = \sqrt{46}$$

## Practice Problems

Problems ??-??

Find the length of the following vectors. Enter your answers in exact, simplified form. (e.g.  $5\sqrt{3}$ )

**Problem 1**

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Answer:

$$\|\mathbf{u}\| = \sqrt{\dots \dots}$$

**Problem 2**

$$\mathbf{v} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$$

Answer: Enter your answer in the most simplified exact form. (e.g.  $a\sqrt{b}$ )

$$\|\mathbf{v}\| = \dots \dots \sqrt{\dots \dots}$$

**Problem 3**

$$\mathbf{w} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

Answer: Enter your answer in the most simplified exact form. (e.g.  $a\sqrt{b}$ )

$$\|\mathbf{w}\| = \dots \dots \sqrt{\dots \dots}$$

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**Problem 4**

$$\mathbf{q} = \begin{bmatrix} -10 \\ 4 \\ -7 \\ 5 \\ 3 \end{bmatrix}$$

Answer:

$$\|\mathbf{q}\| = \sqrt{\dots \dots}$$

**Problem 5** Find the component form of vector  $\mathbf{v}$  in  $\mathbb{R}^2$  if we know that  $\|\mathbf{v}\| = 15$ , the  $x$  component of  $\mathbf{v}$  is  $-9$  and the vector is located in the third quadrant.

Answer:

$$\mathbf{v} = \begin{bmatrix} -9 \\ \dots \dots \end{bmatrix}$$

**Problem 6** For a vector in  $\mathbb{R}^2$  with a length of 26 and the  $y$  component of 10, what are the possibilities for the  $x$  component? List the possibilities in an increasing order.

Answer:

$\dots \dots$ , and  $\dots \dots$

**Problem 7** Let  $\mathbf{v} = \begin{bmatrix} 1 \\ y \\ 4 \end{bmatrix}$ . Find all possible values of  $y$  if  $\|\mathbf{v}\| = 9$ . List the possibilities in an increasing order.

Answer:

$\dots \dots$ , and  $\dots \dots$

**Problem 8** For a vector in  $\mathbb{R}^4$ , what are the possibilities for the fourth component if the length of the vector is 14, and the  $x$ ,  $y$  and  $z$  components are 1, 5 and 13, respectively? List the possibilities in an increasing order.

Answer:

$\dots \dots$ , and  $\dots \dots$

**Problem 9** The points  $A(2, 1, 4)$ ,  $B(4, 2, -1)$ , and  $C(6, 8, 7)$  form a triangle in  $\mathbb{R}^3$ . Is it a right triangle?