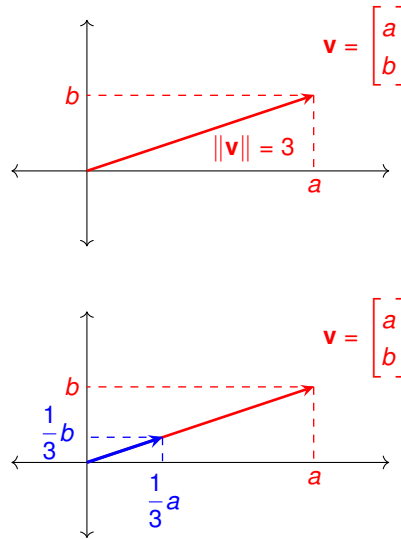


1 Unit Vector in the Direction of a Given Vector

Recall that a *unit* vector is a vector of length 1. Given a non-zero vector $\mathbf{v}$, we can find a unit vector in the same direction by multiplying $\mathbf{v}$ by an appropriate scalar. For example, if $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\|\mathbf{v}\| = 3$, then a unit vector $\mathbf{u}$ in the same direction is given by $\mathbf{u} = \begin{bmatrix} a/3 \\ b/3 \end{bmatrix} = \begin{bmatrix} a/\|\mathbf{v}\| \\ b/\|\mathbf{v}\| \end{bmatrix}$.



In general, dividing a non-zero vector by its own magnitude produces a unit vector in the same direction. We summarize this observation in a theorem.

Theorem 1. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be a non-zero vector in $\mathbb{R}^n$. Vector $\mathbf{u}$ given by

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} = \begin{bmatrix} v_1 / \|\mathbf{v}\| \\ v_2 / \|\mathbf{v}\| \\ \vdots \\ v_n / \|\mathbf{v}\| \end{bmatrix}$$

is a unit vector in the direction of $\mathbf{v}$.

Proof Because $\mathbf{u}$ is a positive scalar multiple of $\mathbf{v}$, $\mathbf{u}$ points in the direction of $\mathbf{v}$. We now show that $\|\mathbf{u}\| = 1$.

$$\begin{aligned}\|\mathbf{u}\| &= \sqrt{\left(\frac{v_1}{\|\mathbf{v}\|}\right)^2 + \left(\frac{v_2}{\|\mathbf{v}\|}\right)^2 + \dots + \left(\frac{v_n}{\|\mathbf{v}\|}\right)^2} \\ &= \frac{1}{\|\mathbf{v}\|} \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} \\ &= \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1\end{aligned}$$

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Example 1. Find a unit vector in the direction of $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Explanation. We first compute $\|\mathbf{v}\|$.

$$\|\mathbf{v}\| = \sqrt{4 + 9 + 1 + 1} = \sqrt{15}$$

By Theorem ??,

$$\mathbf{u} = \begin{bmatrix} 2/\sqrt{15} \\ -3/\sqrt{15} \\ 1/\sqrt{15} \\ 0 \\ 1/\sqrt{15} \end{bmatrix} = \frac{1}{\sqrt{15}} \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Practice Problems

Problem 1 Find a unit vector in the direction of the given vector $\mathbf{v}$.

$$\mathbf{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Answer: $\mathbf{u} = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$

Problem 2 Find a unit vector in the direction of the given vector $\mathbf{v}$.

$$\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

Enter exact answers. No decimal approximations.

Answer: $\mathbf{u} = \begin{bmatrix} \dots \dots \\ \dots \dots \\ \dots \dots \end{bmatrix}$

Problem 3 Let $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ \sqrt{7} \end{bmatrix}$. Apply the concepts from this section to find a vector $\mathbf{w}$ that points in the same direction as $\mathbf{v}$ and whose length is 5.

Enter exact answers. No decimal approximations.

Answer: $\mathbf{w} = \begin{bmatrix} \dots \dots \\ \dots \dots \\ \dots \dots \end{bmatrix}$