
1 Linear Combinations of Vectors

When studying vectors, the two main operations we have learned about are vector addition and scalar multiplication. Both are involved in the important concept of a *linear combination* of vectors.

Definition 1. A vector $\mathbf{v}$ is said to be a *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ if

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n$$

for some scalars $a_1, a_2, \dots, a_n$.

For example, $\begin{bmatrix} -4 \\ 9 \\ -10 \\ -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} -1 \\ 3 \\ -3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ because

$$\begin{bmatrix} -4 \\ 9 \\ -10 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 3 \\ -3 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

In this section we will focus on vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Visualizing Linear Combinations in $\mathbb{R}^2$ and $\mathbb{R}^3$

Let's start by visualizing linear combinations of two vectors in $\mathbb{R}^2$.

Exploration 1 Answer the questions below using the GeoGebra interactive. To use the interactive, you can

1. Change vectors $\mathbf{v}$ and $\mathbf{w}$ by dragging the tips of these vectors.
2. Change the coefficients k_1 and k_2 of the linear combination by using sliders.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

1. Let $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find k_1 and k_2 such that $k_1\mathbf{w} + k_2\mathbf{v} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$$k_1 = \dots\dots, \quad k_2 = \dots\dots$$

2. Let $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$. Find k_1 and k_2 such that $k_1\mathbf{w} + k_2\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$$k_1 = \dots\dots, \quad k_2 = \dots\dots$$

3. Use the same vectors $\mathbf{w}$ and $\mathbf{v}$ as in the previous part. Do you think it is possible to express any vector in $\mathbb{R}^2$ as a linear combination of $\mathbf{w}$ and $\mathbf{v}$? (Yes, / No)

4. Let $\mathbf{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$. Do you think it is possible to express any vector in $\mathbb{R}^2$ as a linear combination of $\mathbf{w}$ and $\mathbf{v}$? (Yes, / No)

Visualizing linear combinations of vectors in $\mathbb{R}^3$ is more difficult than doing so in $\mathbb{R}^2$. The following Exploration will help you do this.

Exploration 2 We will start by visualizing linear combinations of two vectors, $\mathbf{u}$ and $\mathbf{v}$, in $\mathbb{R}^3$. To use the interactive below, define vectors $\mathbf{u}$ and $\mathbf{v}$. Use sliders to change the coefficients k_1 and k_2 of the linear combination. You will see the linear combination $k_1\mathbf{u} + k_2\mathbf{v}$ as the pink vector along the diagonal of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

RIGHT-CLICK and DRAG the left panel to rotate the image.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

We will now consider three vectors. Define vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Use sliders to change the coefficients k_1 , k_2 and k_3 of the linear combination. The linear combination $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w}$ is shown as the pink vector along the diagonal of the parallelepiped.

RIGHT-CLICK and DRAG the left panel to rotate the image.

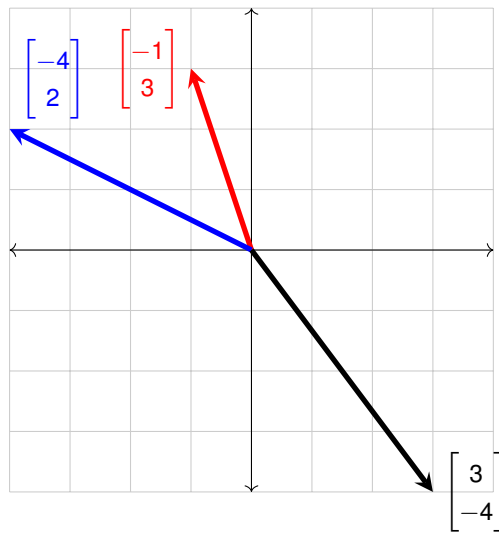
Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

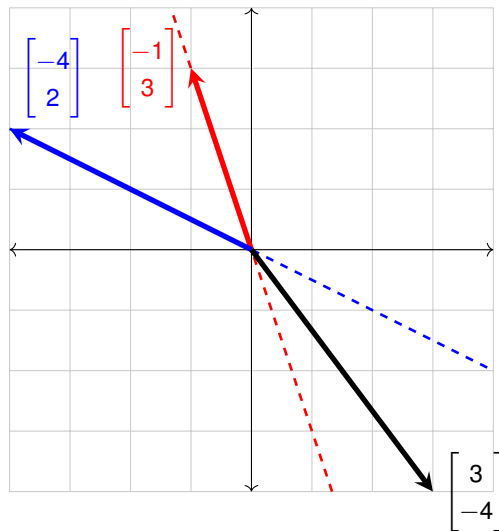
Geometry of Linear Combinations

Example 1. Use geometry to express $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

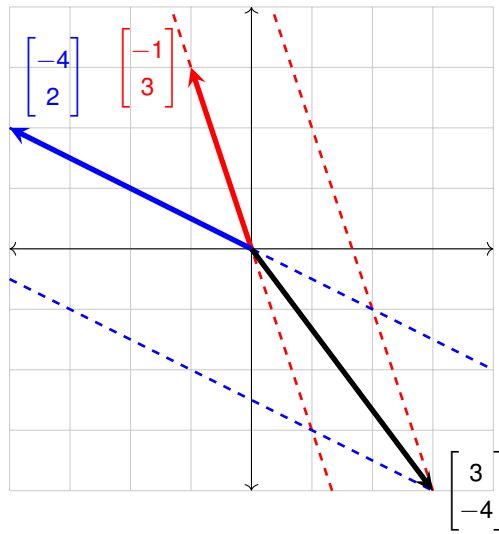
Explanation. We are looking for $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ to be the diagonal of a parallelogram determined by scalar multiples of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$.



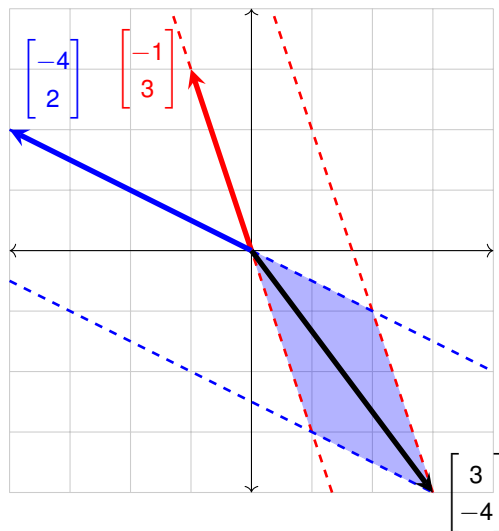
Because a scalar multiple of a vector can point in the same direction as the vector or in the opposite direction, we will start by drawing straight lines determined by the two vectors.



The two lines that we drew will contain the sides of the parallelogram we are looking for. To find the other two sides we will draw lines parallel to $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ through the head of vector $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

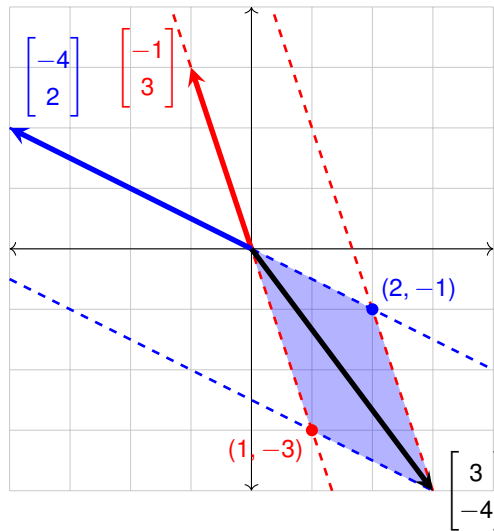


Now the parallelogram is clearly visible.



The last remaining task is to identify the sides of the parallelogram as scalar multiples of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$. Observe

that vectors $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ determine the parallelogram.



Vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is half the length of $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and points in the opposite direction, while the vector $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is the same length as $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and also points in the opposite direction.

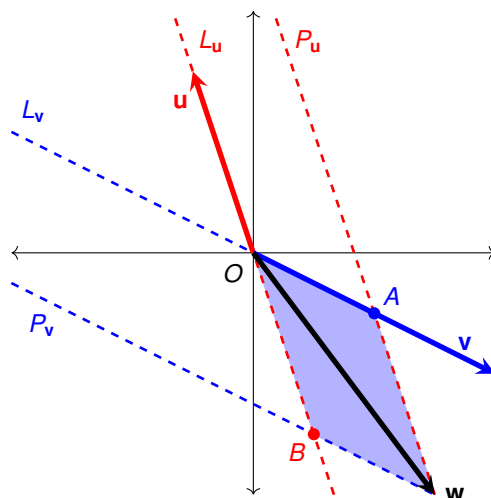
We write $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ as follows

$$\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 3 \end{bmatrix} + \left(-\frac{1}{2}\right) \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

The method we used in Example ?? to express the given vector as a linear combination of two other vectors is sufficiently useful that we summarize the steps.

Procedure 1. Given two non-collinear vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, and a vector $\mathbf{w}$, we can express $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$ as follows:

1. Draw lines $L_{\mathbf{u}}$ and $L_{\mathbf{v}}$ determined by $\mathbf{u}$ and $\mathbf{v}$, respectively.
2. Through the head of vector $\mathbf{w}$, draw lines $P_{\mathbf{u}}$ and $P_{\mathbf{v}}$, parallel to $L_{\mathbf{u}}$ and $L_{\mathbf{v}}$, respectively.
3. Let A be the point of intersection of $P_{\mathbf{u}}$ and $L_{\mathbf{v}}$.
4. Let B be the point of intersection of $P_{\mathbf{v}}$ and $L_{\mathbf{u}}$.
5. Let O denote the origin. Then $\overrightarrow{OA} = k_1 \mathbf{v}$ and $\overrightarrow{OB} = k_2 \mathbf{u}$ for some scalars k_1 and k_2 .
6. We have $\mathbf{w} = k_2 \mathbf{u} + k_1 \mathbf{v}$.



Exploration 3 This GeoGebra interactive will allow you to go through the steps given in Procedure ?? for a combination of vectors of your choice. To use the interactive

1. Enter components of vectors $\mathbf{u}$ and $\mathbf{v}$.
2. Enter components of vector $\mathbf{w}$ that you want to express as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.
3. Use the navigation bar to go through the steps of Procedure ??

Access GeoGebra interactives through the online version of this text at

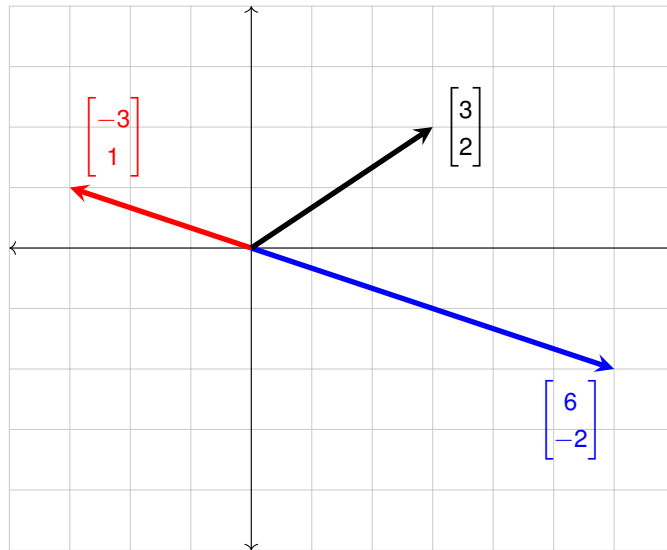
<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

From Geometry to Algebra of Linear Combinations

One of the stipulations in Procedure ?? is that vectors $\mathbf{u}$ and $\mathbf{v}$ should be non-collinear. You can use the interactive in Exploration ?? to investigate what happens when $\mathbf{u}$ and $\mathbf{v}$ are collinear. The following example examines what happens from a geometric as well as an algebraic standpoint.

Example 2. Can the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ be written as a linear combination of vectors $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$?

Explanation. We will start with a geometric approach.



Observe that $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$ are scalar multiples of each other and lie on the same line.

A linear combination of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$ has the form:

$$a \begin{bmatrix} 6 \\ -2 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \end{bmatrix} = a(-2) \begin{bmatrix} -3 \\ 1 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \end{bmatrix} = (-2a + b) \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

This shows that all linear combinations of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$ will be scalar multiples of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, and therefore lie on the same line as $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Since $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ does not lie on the line determined by $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ it cannot be expressed as a linear combination of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

We can also address this question algebraically. To express $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$, we need to solve the equation.

$$a \begin{bmatrix} -3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

This gives us a system of equations

$$\begin{aligned} -3a + 6b &= 3 \\ a - 2b &= 2 \end{aligned}$$

When you try to solve this system, you will find that the system is inconsistent. Thus, $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ cannot be written as a linear combination of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

We know that there is no way to express $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as a linear combination of vectors $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$. What would happen if we tried to apply Procedure ?? to these vectors? You can use the GeoGebra interactive in Exploration ?? to find out.

Example 3. Express $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$. Interpret your results geometrically.

Explanation. We need to find scalars a and b such that

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

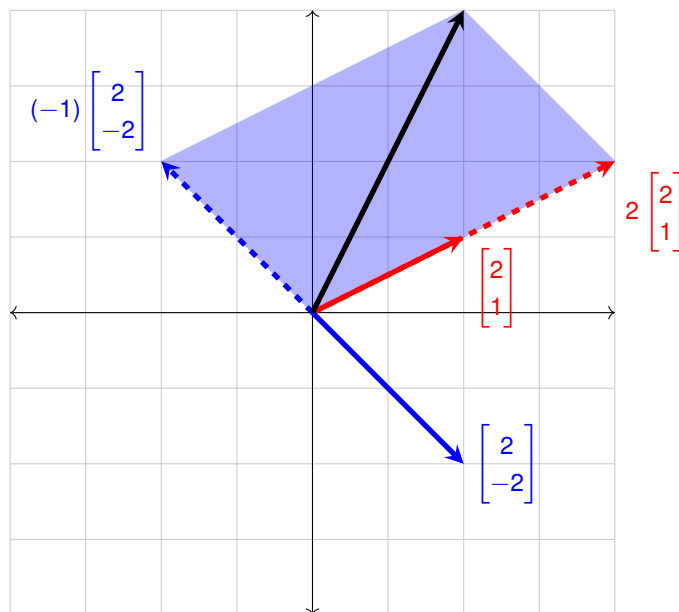
This amounts to solving a system of linear equations

$$\begin{aligned} 2a + 2b &= 2 \\ a - 2b &= 4 \end{aligned}$$

Use your favorite method to solve this system. (Hint: adding the second equation to the first will work well for this system.) You will find that $a = 2$ and $b = -1$. Now we can write $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ as follows:

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Geometrically speaking, this means that the vector $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is the diagonal of the parallelogram determined by $2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $(-1) \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. The original vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ are shown below together with the parallelogram and its diagonal.



Example 4. If possible, express $\begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$.

Explanation. We are looking for coefficients a and b such that

$$a \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$$

This translates into a system of equations

$$\begin{aligned} a + 3b &= 7 \\ -2a &= 4 \\ a - b &= -5 \end{aligned}$$

Solve this system for a and b , and enter your answers below:

$$a = \dots\dots \text{ and } b = \dots\dots$$

We conclude that $\begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, and write:

$$\begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} = \dots\dots \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \dots\dots \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

Example 5. Set up a system of equations that can be used to express $\begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 4 \\ -2 \end{bmatrix}$, $\begin{bmatrix} -2 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 4 \\ -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 4 \end{bmatrix}$, or to determine that such a combination does not exist. Do not solve the system.

Explanation. We are looking for x_1, x_2, x_3 and x_4 such that

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 4 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -1 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 4 \\ -3 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

This translates into the following system of equations:

$$\begin{array}{ccccccccc} x_1 & - & 2x_2 & & & + & x_4 & = & 2 \\ & & -x_2 & + & 4x_3 & + & x_4 & = & -1 \\ 4x_1 & + & x_2 & - & 3x_3 & - & x_4 & = & 3 \\ -2x_1 & - & x_2 & + & x_3 & + & 4x_4 & = & 0 \end{array}$$

Practice Problems

Problem 4 Solve a system of linear equations to express $\begin{bmatrix} -1 \\ 7 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

System of linear equations:

$$\begin{array}{ccccccc} \dots\dots a & + & \dots\dots b & = & \dots\dots \\ \dots\dots a & + & \dots\dots b & = & \dots\dots \end{array}$$

Values of a and b :

$$a = \dots\dots \text{ and } b = \dots\dots$$

Linear Combination:

$$\begin{bmatrix} -1 \\ 7 \end{bmatrix} = \dots\dots \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \dots\dots \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Problem 5 Use Procedure ?? to express $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Linear Combination:

$$\begin{bmatrix} -3 \\ 0 \end{bmatrix} = \dots\dots \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \dots\dots \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Problem 6 Use two different approaches (algebraic and geometric) to explain why the vector $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ cannot be expressed as a linear combination of vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

Problem 7 We have seen Procedure ?? applied to vectors in $\mathbb{R}^2$. The same process can, in certain cases be applied to vectors in $\mathbb{R}^3$. Follow the steps in Procedure ?? to express one vector as a linear combination of two given vectors.

The following GeoGebra interactive shows vectors $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$. Use the navigation bar to click through the construction steps. RIGHT-CLICK and DRAG to rotate the image.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

1. Can $\mathbf{w}$ be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$?

Multiple Choice:

- (i) No, because $\mathbf{w}$ is not between $\mathbf{u}$ and $\mathbf{v}$.
 - (ii) Yes, because all three vectors are in the same plane.
 - (iii) Yes, because all three vectors are in the same plane, AND $\mathbf{u}$ and $\mathbf{v}$ are not collinear.
2. Use the navigation bar at the bottom of the interactive window to view construction steps of Procedure ?? applied to vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. (Right-click and drag to rotate the image.) Use the final image to express $\mathbf{w}$ as a linear combination of $\mathbf{v}$ (blue) and $\mathbf{u}$ (red).

$$\mathbf{w} = \dots \mathbf{v} + \dots \mathbf{u}$$

Problem 8 The following GeoGebra interactive shows vectors $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 7 \\ 4 \\ 0 \end{bmatrix}$. RIGHT-

CLICK and DRAG to rotate the image. Use geometry to explain why $\mathbf{w}$ cannot be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{u}$.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

We can also show that $\mathbf{w}$ is not a linear combination of $\mathbf{v}$ and $\mathbf{u}$ algebraically by attempting to solve a system of equations corresponding to

$$x_1 \mathbf{v} + x_2 \mathbf{u} = \mathbf{w}$$

Set up the system of equations

$$\begin{array}{rcl} \dots x_1 & + & \dots x_2 = \dots \\ \dots x_1 & + & \dots x_2 = \dots \\ \dots x_1 & + & \dots x_2 = \dots \end{array}$$

Find the reduced row echelon form.

$$\left[\begin{array}{cc|c} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{array} \right]$$