
1 Dot Product and its Properties

Definition 1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. The *dot product* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \cdot \mathbf{v}$, is given by

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n.$$

Example 1. Find $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{u} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$.

Explanation.

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = (-2)(3) + (0)(2) + (1)(-4) = -6 - 4 = -10$$

Note that the dot product of two vectors is a scalar. For this reason, the dot product is sometimes called a *scalar product*.

Properties of the Dot Product

A quick examination of Example ?? will convince you that the dot product is *commutative*. In other words, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. This and other properties of the dot product are stated below.

Theorem 1. The following properties hold for vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ and scalar k in $\mathbb{R}$.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$
5. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
6. $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}$

We will prove Property ?. The remaining properties are left as exercises.

Proof of Property ??:

$$\begin{aligned}
 (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= \left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \\
 &= (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + \dots + (u_n + v_n)w_n \\
 &= u_1 w_1 + v_1 w_1 + u_2 w_2 + v_2 w_2 + \dots + u_n w_n + v_n w_n \\
 &= (u_1 w_1 + u_2 w_2 + \dots + u_n w_n) + (v_1 w_1 + v_2 w_2 + \dots + v_n w_n) \\
 &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}
 \end{aligned}$$

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We will illustrate Property ?? with an example.

Example 2. Let $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Use $\mathbf{u}$ to illustrate Property ?? of Theorem ??.

Explanation.

$$\|\mathbf{u}\|^2 = (-2)^2 + 3^2 = (-2)(-2) + (3)(3) = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \mathbf{u} \cdot \mathbf{u}$$

If we take the square root of both sides of the equation in Property ??, we get an alternative way to think of the [length of a vector](#).

Corollary 1 (Length of a Vector). Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$, then the *length*, or the *magnitude*, of $\mathbf{v}$ is given by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

Practice Problems

Problem 1 Find the dot product of $\mathbf{u}$ and $\mathbf{v}$ if

$$\mathbf{u} = \begin{bmatrix} -1 \\ -2 \\ 5 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ -3 \\ 1 \end{bmatrix}$$

Answer:

$$\mathbf{u} \cdot \mathbf{v} = \dots$$

Problem 2 Find the dot product of $\mathbf{u}$ and $\mathbf{v}$ if

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Answer:

$$\mathbf{u} \cdot \mathbf{v} = \dots \dots$$

Problem 3 Use vector $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix}$ to illustrate Property ?? of Theorem ??.

Problem 4 Prove Properties ??, ??, ??, ?? and ?? of Theorem ??.

Problem 5 From the given list of vector pairs, identify ALL pairs of vectors that lie on perpendicular lines.

Select All Correct Answers:

1. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

3. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

4. $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

Compute $\mathbf{u} \cdot \mathbf{v}$ for each pair. What do you observe?

Problem 6 1. Vector $\mathbf{u}$ parallel to the line $y = mx$ has the form $\mathbf{u} = k \begin{bmatrix} 1 \\ m \end{bmatrix}$. Assuming that $m \neq 0$, find the general form for a vector $\mathbf{v}$ perpendicular to $y = mx$.

2. Find $\mathbf{u} \cdot \mathbf{v}$.

3. What is true about the dot product of perpendicular vectors?