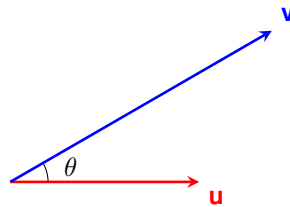

1 Dot Product and Angle

Given two vectors $\mathbf{u}$ and $\mathbf{v}$, let θ be the angle between them such that $0 \leq \theta \leq \pi$. We will refer to θ as the *included angle*.

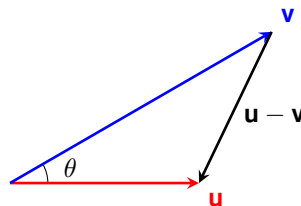


The following theorem establishes a relationship between the dot product and the included angle.

Theorem 1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$, and let θ be the included angle. Then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

Proof Consider the triangle formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.



By the Law of Cosines we have:

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2 \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

By Theorem [????](#) of [Dot Product and its Properties](#)

$$(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2 \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

By Theorem [????](#) of [Dot Product and its Properties](#)

$$(\mathbf{u} - \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} - \mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2 \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

By Theorem [????](#) of [Dot Product and its Properties](#)

$$\mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2 \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

By Theorem [????](#) of [Dot Product and its Properties](#)

$$\begin{aligned} -2(\mathbf{u} \cdot \mathbf{v}) &= -2 \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\ \mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \end{aligned}$$

■

Example 1. Find the included angle between vectors $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Explanation. By Theorem ??, $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$.

$$\mathbf{u} \cdot \mathbf{v} = 2 - 2 - 12 = -12$$

$$\|\mathbf{u}\| = \sqrt{4 + 1 + 16} = \sqrt{21}$$

$$\|\mathbf{v}\| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$\cos \theta = \frac{-12}{\sqrt{21}\sqrt{14}}$$

$$\theta \approx 134.4^\circ$$

Orthogonal Vectors

Definition 1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. We say $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.

We can use Theorem ?? to show that two non-zero *orthogonal* vectors of $\mathbb{R}^n$ are simply *perpendicular* vectors (the included angle is 90°). To see this, suppose that $\mathbf{u} \cdot \mathbf{v} = 0$ for nonzero vectors $\mathbf{u}, \mathbf{v}$. Then from Theorem ?? we have

$$0 = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

Since $\mathbf{u}, \mathbf{v}$ are nonzero vectors, we have $0 = \cos \theta$, which implies $\theta = 90^\circ$. The converse also holds. If $\theta = 90^\circ$, then the dot product is clearly 0.

The reason we prefer the term “orthogonal” to “perpendicular” in this course is because $\mathbb{R}^n$ is only one example of a *vector space*, and the dot product is only one example of a more general product, called an *inner product*. For vectors in $\mathbb{R}^n$ a zero dot product happens to coincide with the geometric idea of perpendicularity, but there are many vector spaces that do not possess the visual geometry of $\mathbb{R}^n$. (Later in the text, you will encounter vector spaces whose vectors are polynomial functions!) In these more abstract settings, a zero inner product still signals a special relationship between vectors. The term *orthogonal* captures this relationship.

Remark 1. By our definition, the zero vector is orthogonal to any vector. However, we will not use the word *perpendicular* when the zero vector is involved, as it is not possible to talk about an “included angle”.

Practice Problems

Problem 1 Find the degree measure of the included angle, θ for each pair of vectors. Round your answers to the nearest tenth.

1. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$.

Answer: $\theta = \dots\dots^\circ$

2. $\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

Answer: $\theta = \dots\dots^\circ$

3. $\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ -2 \\ 4 \end{bmatrix}$

Answer: $\theta = \dots\dots^\circ$

Problem 2 What does the sign of the dot product tell us about the included angle?

Problem 3 Find all values of a so that $\begin{bmatrix} a^2 \\ 2a \\ 1 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. List your answers in increasing order.

Answer: $\dots\dots, \dots\dots$

Problem 4 Find the value of x for which the vector $\begin{bmatrix} x \\ -4 \end{bmatrix}$ is parallel to the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. What is the measure of the included angle, θ ? Find the measure of the included angle using Theorem ???. Do the two results agree?

Answer:

$$x = \dots\dots$$

$$\theta = \dots\dots^\circ$$

Problem 5 Prove that if $\mathbf{u}$ is a unit vector, then $\mathbf{u} \cdot \mathbf{u} = 1$.

Problem 6 Prove that if $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors, then $-1 \leq \mathbf{u}_1 \cdot \mathbf{u}_2 \leq 1$. In what cases are the extreme values of 1 and -1 attained?

Problem 7 Imagine a clock with hands represented by vectors $\mathbf{m}$ and $\mathbf{h}$, as shown below. At what whole hour will $\mathbf{m} \cdot \mathbf{h}$ attain its maximum value? At what whole hour will $\mathbf{m} \cdot \mathbf{h}$ be as small as possible?



Answer:

$m \cdot h$ is greatest at : 00 o'clock

$m \cdot h$ is smallest at : 00 o'clock

Photo Credits

The following images are courtesy of [Wikimedia Commons](#)

Hannes Grobe, *Wall clock manufactured by Telefonbau & Normalzeit*. CC-BY 3.0