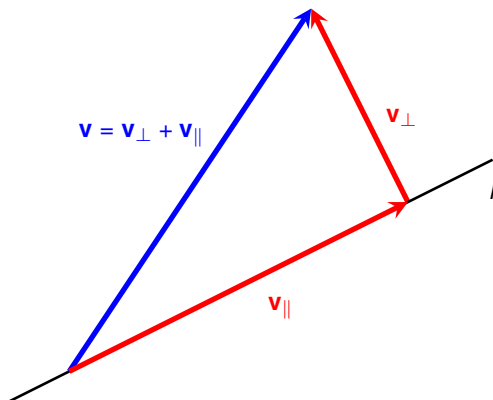

1 Orthogonal Projections

Given a line l and a vector $\mathbf{v}$ emanating from a point on l , it is sometimes convenient to express $\mathbf{v}$ as the sum of a vector $\mathbf{v}_{\parallel}$, parallel to l , and a vector $\mathbf{v}_{\perp}$, perpendicular to l . If you have taken a physics course, you may have seen a force vector decomposed into the sum of two components: one parallel and one perpendicular to the direction of motion.



Suppose $\mathbf{d}$ is a direction vector for l . Then $\mathbf{v}_{\parallel} = k\mathbf{d}$ for some scalar k . Our goal is to find k .

$$\begin{aligned}\mathbf{v} \cdot \mathbf{d} &= (\mathbf{v}_{\parallel} + \mathbf{v}_{\perp}) \cdot \mathbf{d} \\ &= (k\mathbf{d} + \mathbf{v}_{\perp}) \cdot \mathbf{d} \\ &= k\mathbf{d} \cdot \mathbf{d} + \mathbf{v}_{\perp} \cdot \mathbf{d} \\ &= k\|\mathbf{d}\|^2 + 0 \\ &= k\|\mathbf{d}\|^2\end{aligned}$$

We conclude that

$$k = \frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}$$

and

$$\mathbf{v}_{\parallel} = k\mathbf{d} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d}$$

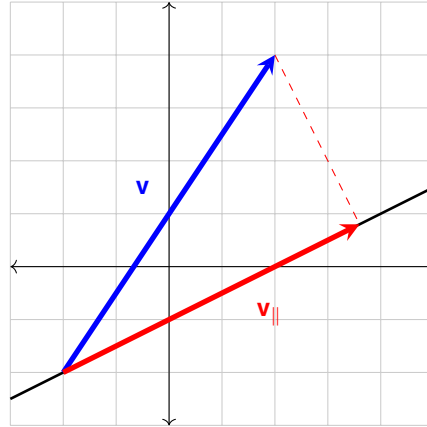
The vector $\mathbf{v}_{\parallel} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d}$ is called the *projection of $\mathbf{v}$ onto $\mathbf{d}$* . In our discussion, $\mathbf{d}$ is a direction vector for line l . So, we can also say that $\mathbf{v}_{\parallel}$ is the *projection of $\mathbf{v}$ onto l* .

To find $\mathbf{v}_{\perp}$, observe that $\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel}$.

Definition 1. Let $\mathbf{v}$ be a vector, and let $\mathbf{d}$ be a non-zero vector. The *projection of $\mathbf{v}$ onto $\mathbf{d}$* is given by

$$\text{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d}$$

Example 1. Find the projection of $\mathbf{v}$, shown below, onto the line given by $y = \frac{1}{2}x - 1$.



Explanation. We begin by finding vectors $\mathbf{v}$ and $\mathbf{d}$. The tail of $\mathbf{v}$ is located at $(-2, -2)$, and the head of $\mathbf{v}$ is at $(2, 4)$. Using the “head-tail” formula we get

$$\mathbf{v} = \begin{bmatrix} 2 - (-2) \\ 4 - (-2) \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

The direction vector for the line $y = \frac{1}{2}x - 1$ is

$$\mathbf{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We find that $\mathbf{v} \cdot \mathbf{d} = 14$ and $\|\mathbf{d}\|^2 = 5$. Thus

$$\text{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d} = \frac{14}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 28/5 \\ 14/5 \end{bmatrix}$$

Distance from a Point to a Line

The shortest distance from a point to a line is the length of the perpendicular line segment dropped from the point to the line. Vector projection formula will help us find the length of such a perpendicular.

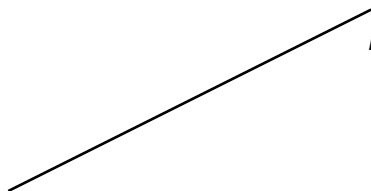
Example 2. Let $A(2, -1, 1)$ be a point in $\mathbb{R}^3$. Suppose line l is given by parametric equations

$$x = t + 3$$

$$y = -t + 1$$

$$z = t - 2$$

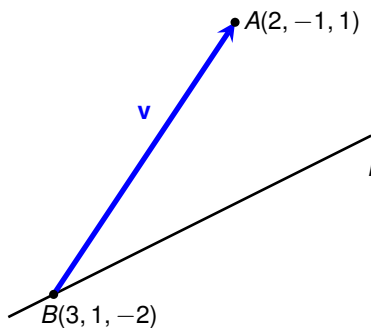
$$\bullet A(2, -1, 1)$$



Find the distance from A to l .

Explanation. We will first construct a vector $\mathbf{v}$ by picking an arbitrary point B on l to be the tail of $\mathbf{v}$ and using point A as the head of $\mathbf{v}$. An easy point to choose on line l is the point $(3, 1, -2)$ that corresponds to $t = 0$. Now we have

$$\mathbf{v} = \overrightarrow{BA} = \begin{bmatrix} 2 - 3 \\ -1 - 1 \\ 1 - (-2) \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

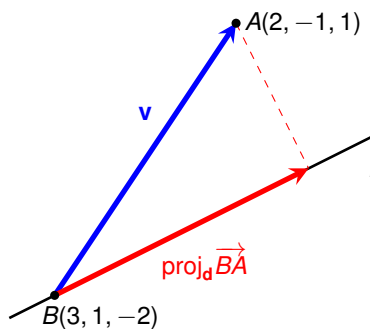


The line has a direction vector

$$\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

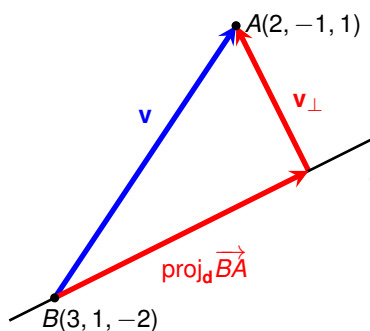
We will now find the projection of $\overrightarrow{BA}$ onto l

$$\text{proj}_{\mathbf{d}} \overrightarrow{BA} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d} = \frac{4}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -4/3 \\ 4/3 \end{bmatrix}$$



Next, we find $\mathbf{v}_\perp$.

$$\mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4/3 \\ -4/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} -7/3 \\ -2/3 \\ 5/3 \end{bmatrix}$$



Finally, to find the distance between point A and line l , we find the magnitude of $\mathbf{v}_\perp$.

$$\|\mathbf{v}_\perp\| = \frac{1}{3}\sqrt{49 + 4 + 25} = \frac{\sqrt{78}}{3}$$

Practice Problems

Problems ??-??

Find $\text{proj}_d \mathbf{v}$.

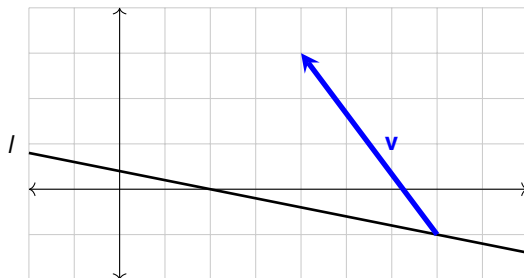
Problem 1 If $\mathbf{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ then

$$\text{proj}_d \mathbf{v} = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

Problem 2 If $\mathbf{d} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}$ then

$$\text{proj}_{\mathbf{d}} \mathbf{v} = \begin{bmatrix} \dots\dots \\ \dots\dots \\ \dots\dots \end{bmatrix}$$

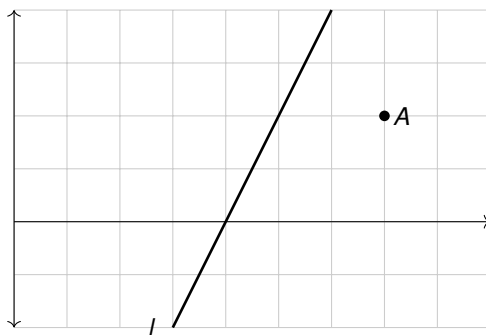
Problem 3 Find the projection of vector $\mathbf{v}$ onto line l . (If entering answers in decimal form, round to the nearest one hundredth.)



Answer:

$$\begin{bmatrix} \dots\dots \\ \dots\dots \end{bmatrix}$$

Problem 4 Find the distance between point A and line l .



Answer: $\sqrt{\dots\dots}$

Problem 5 Show that $\text{proj}_{\mathbf{d}} \mathbf{v}$ does not depend on the length of $\mathbf{d}$ by proving that $\text{proj}_{\mathbf{d}} \mathbf{v} = \text{proj}_{k\mathbf{d}} \mathbf{v}$ for $k \neq 0$. What does this result mean geometrically? Illustrate your response with a diagram.

Problem 6 Find the radius of a circle centered at $(4, 2)$ if the line $y = \frac{3}{2}x + 3$ is tangent to the circle. Enter your response as a fraction.

Answer:

$$r = \sqrt{\dots\dots}$$

The graph below shows the line $y = \frac{3}{2}x + 3$ together with a circle of radius 1. Change the value of r to the radius you have found to visualize the correct answer.

Access GeoGebra interactives through the online version of this text at <https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.