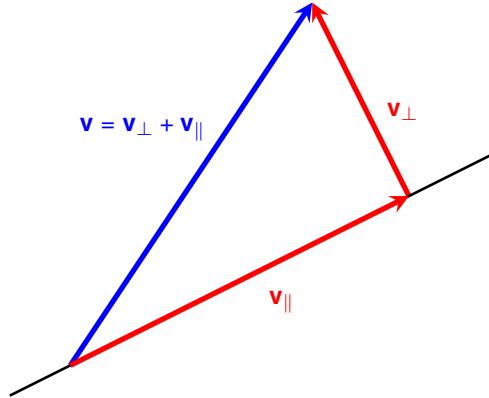

1 Orthogonal Projections

Given a line l and a vector $\mathbf{v}$ emanating from a point on l , it is sometimes convenient to express $\mathbf{v}$ as the sum of a vector $\mathbf{v}_{\parallel}$, parallel to l , and a vector $\mathbf{v}_{\perp}$, perpendicular to l . If you have taken a physics course, you may have seen a force vector decomposed into the sum of two components: one parallel and one perpendicular to the direction of motion.



Suppose $\mathbf{d}$ is a direction vector for l . Then $\mathbf{v}_{\parallel} = k\mathbf{d}$ for some scalar k . Our goal is to find k .

$$\begin{aligned}\mathbf{v} \cdot \mathbf{d} &= (\mathbf{v}_{\parallel} + \mathbf{v}_{\perp}) \cdot \mathbf{d} \\ &= (k\mathbf{d} + \mathbf{v}_{\perp}) \cdot \mathbf{d} \\ &= k\mathbf{d} \cdot \mathbf{d} + \mathbf{v}_{\perp} \cdot \mathbf{d} \\ &= k\|\mathbf{d}\|^2 + 0 \\ &= k\|\mathbf{d}\|^2\end{aligned}$$

We conclude that

$$k = \frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}$$

and

$$\mathbf{v}_{\parallel} = k\mathbf{d} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d}$$

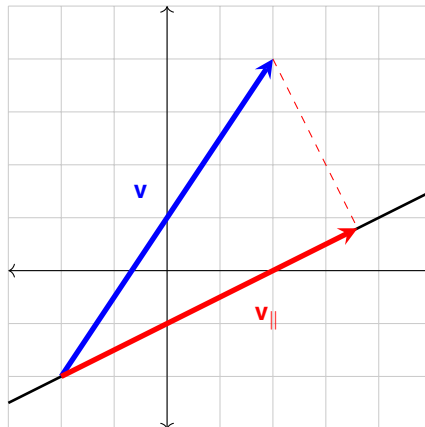
The vector $\mathbf{v}_{\parallel} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d}$ is called the *projection of $\mathbf{v}$ onto $\mathbf{d}$* . In our discussion, $\mathbf{d}$ is a direction vector for line l . So, we can also say that $\mathbf{v}_{\parallel}$ is the *projection of $\mathbf{v}$ onto l* .

To find $\mathbf{v}_{\perp}$, observe that $\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel}$.

Definition 1. Let $\mathbf{v}$ be a vector, and let $\mathbf{d}$ be a non-zero vector. The *projection of $\mathbf{v}$ onto $\mathbf{d}$* is given by

$$\text{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d}$$

Example 1. Find the projection of $\mathbf{v}$, shown below, onto the line given by $y = \frac{1}{2}x - 1$.



Explanation. We begin by finding vectors $\mathbf{v}$ and $\mathbf{d}$. The tail of $\mathbf{v}$ is located at $(-2, -2)$, and the head of $\mathbf{v}$ is at $(2, 4)$. Using the “head-tail” formula we get

$$\mathbf{v} = \begin{bmatrix} 2 - (-2) \\ 4 - (-2) \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

The direction vector for the line $y = \frac{1}{2}x - 1$ is

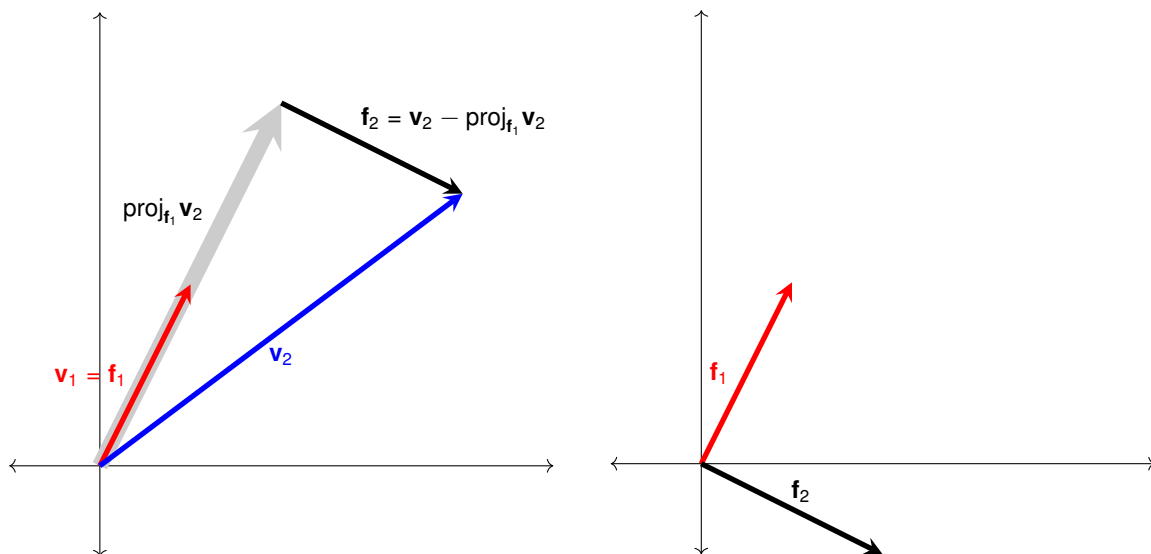
$$\mathbf{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We find that $\mathbf{v} \cdot \mathbf{d} = 14$ and $\|\mathbf{d}\|^2 = 5$. Thus

$$\text{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d} = \frac{14}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 28/5 \\ 14/5 \end{bmatrix}$$

A Visual Guide to Creating an Orthogonal Basis

Given an arbitrary basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{R}^2$, let's start building our orthogonal basis, $\{\mathbf{f}_1, \mathbf{f}_2\}$, by setting $\mathbf{f}_1 = \mathbf{v}_1$. To find the next element of our orthogonal basis, consider the orthogonal projection of $\mathbf{v}_2$ onto $\mathbf{f}_1$. (See the figure below.)



Next, let $\mathbf{f}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{f}_1} \mathbf{v}_2$. Observe that $\mathbf{f}_2$ is orthogonal to $\mathbf{f}_1$ (See Theorem ?? of [Orthogonality and Projections](#)). This gives us an orthogonal collection $\mathcal{B} = \{\mathbf{f}_1, \mathbf{f}_2\}$. It is intuitively clear that $\mathbf{f}_1$ and $\mathbf{f}_2$ are linearly independent. Therefore $\mathcal{B}$ is an orthogonal basis of $\mathbb{R}^2$.

The following exploration illustrates this process dynamically.

Exploration 1 Choose an arbitrary basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{R}^2$ by dragging the tips of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ to desired positions. Use the navigation bar at the bottom of the interactive window to go through the steps of constructing an orthogonal basis of $\mathbb{R}^2$.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

We can apply this process to any two-dimensional subset of $\mathbb{R}^n$. The following exploration will guide you through the process of constructing an orthogonal basis for a plane spanned by two arbitrary vectors in $\mathbb{R}^3$.

Exploration 2 Let $W = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$. W is a plane through the origin in $\mathbb{R}^3$. Use the navigation bar at the bottom of the interactive window to go through the steps of constructing an orthogonal basis for W . RIGHT-CLICK and DRAG to rotate the image for a better view.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

In the next exploration, we take the process of constructing an orthogonal basis to the edge of the visual realm and construct an orthogonal basis for $\mathbb{R}^3$.

Exploration 3 In the GeoGebra interactive below $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$. Use check boxes to go through the steps for constructing an orthogonal basis starting with the given basis. RIGHT-CLICK and DRAG to rotate the image for a better view.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

What we saw above are low-dimension examples of what is known as *Gram Schmidt Orthogonalization*.

Practice Problems

Problem 4 Find $\text{proj}_{\mathbf{d}}\mathbf{v}$ if $\mathbf{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

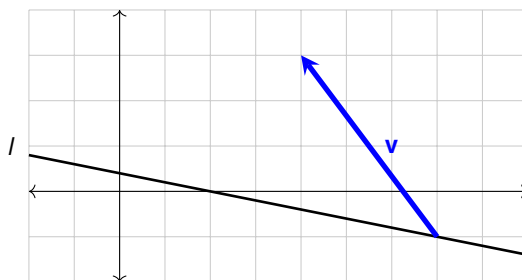
$$\text{proj}_{\mathbf{d}}\mathbf{v} = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

Problem 5 Find $\text{proj}_{\mathbf{d}}\mathbf{v}$ if $\mathbf{d} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}$.

$$\text{proj}_{\mathbf{d}}\mathbf{v} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$

Problem 6 Show that $\text{proj}_{\mathbf{d}}\mathbf{v}$ does not depend on the length of $\mathbf{d}$ by proving that $\text{proj}_{\mathbf{d}}\mathbf{v} = \text{proj}_{k\mathbf{d}}\mathbf{v}$ for $k \neq 0$. What does this result mean geometrically? Illustrate your response with a diagram.

Problem 7 Find the projection of vector $\mathbf{v}$ onto line l . (If entering answers in decimal form, round to the nearest one hundredth.)



Answer:

$$\begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

Problem 8 Convert the given basis $\mathcal{B}$ of $\mathbb{R}^2$ to an orthogonal basis by following the procedure described in this section with $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} \dots\dots \\ \dots\dots \end{bmatrix} \right\}$$