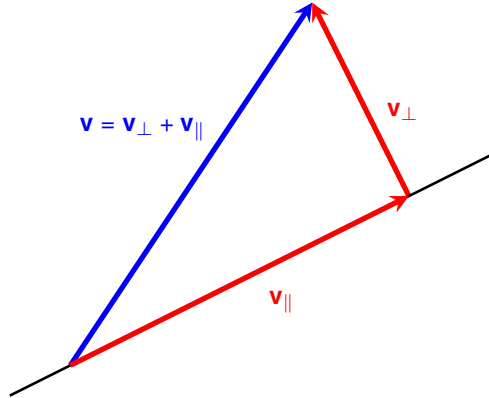

1 Orthogonal Projections

Given a line l and a vector $\mathbf{v}$ emanating from a point on l , it is sometimes convenient to express $\mathbf{v}$ as the sum of a vector $\mathbf{v}_{\parallel}$, parallel to l , and a vector $\mathbf{v}_{\perp}$, perpendicular to l . If you have taken a physics course, you may have seen a force vector decomposed into the sum of two components: one parallel and one perpendicular to the direction of motion.



Suppose $\mathbf{d}$ is a direction vector for l . Then $\mathbf{v}_{\parallel} = k\mathbf{d}$ for some scalar k . Our goal is to find k .

$$\begin{aligned}\mathbf{v} \cdot \mathbf{d} &= (\mathbf{v}_{\parallel} + \mathbf{v}_{\perp}) \cdot \mathbf{d} \\ &= (k\mathbf{d} + \mathbf{v}_{\perp}) \cdot \mathbf{d} \\ &= k\mathbf{d} \cdot \mathbf{d} + \mathbf{v}_{\perp} \cdot \mathbf{d} \\ &= k\|\mathbf{d}\|^2 + 0 \\ &= k\|\mathbf{d}\|^2\end{aligned}$$

We conclude that

$$k = \frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}$$

and

$$\mathbf{v}_{\parallel} = k\mathbf{d} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d}$$

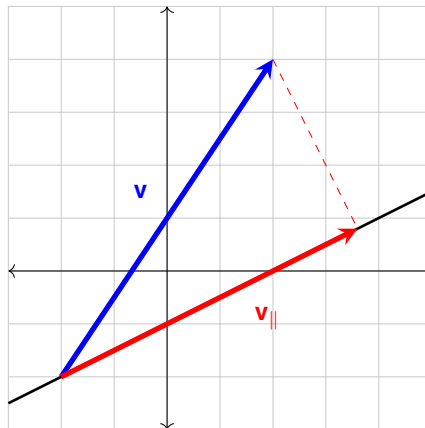
The vector $\mathbf{v}_{\parallel} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d}$ is called the *projection of $\mathbf{v}$ onto $\mathbf{d}$* . In our discussion, $\mathbf{d}$ is a direction vector for line l . So, we can also say that $\mathbf{v}_{\parallel}$ is the *projection of $\mathbf{v}$ onto l* .

To find $\mathbf{v}_{\perp}$, observe that $\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel}$.

Definition 1. Let $\mathbf{v}$ be a vector, and let $\mathbf{d}$ be a non-zero vector. The *projection of $\mathbf{v}$ onto $\mathbf{d}$* is given by

$$\text{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d}$$

Example 1. Find the projection of $\mathbf{v}$, shown below, onto the line given by $y = \frac{1}{2}x - 1$.



Explanation. We begin by finding vectors $\mathbf{v}$ and $\mathbf{d}$. The tail of $\mathbf{v}$ is located at $(-2, -2)$, and the head of $\mathbf{v}$ is at $(2, 4)$. Using the “head-tail” formula we get

$$\mathbf{v} = \begin{bmatrix} 2 - (-2) \\ 4 - (-2) \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

The direction vector for the line $y = \frac{1}{2}x - 1$ is

$$\mathbf{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We find that $\mathbf{v} \cdot \mathbf{d} = 14$ and $\|\mathbf{d}\|^2 = 5$. Thus

$$\text{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \right) \mathbf{d} = \frac{14}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 28/5 \\ 14/5 \end{bmatrix}$$

Practice Problems

Problem 1 Find $\text{proj}_{\mathbf{d}} \mathbf{v}$ if $\mathbf{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

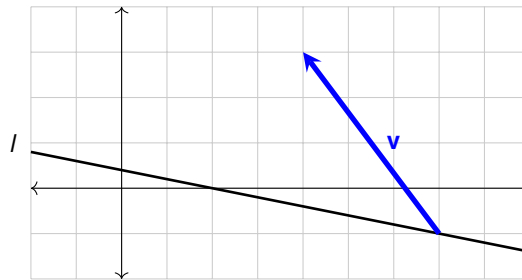
$$\text{proj}_{\mathbf{d}} \mathbf{v} = \begin{bmatrix} \dots\dots \\ \dots\dots \end{bmatrix}$$

Problem 2 Find $\text{proj}_{\mathbf{d}} \mathbf{v}$ if $\mathbf{d} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}$.

$$\text{proj}_{\mathbf{d}} \mathbf{v} = \begin{bmatrix} \dots\dots \\ \dots\dots \\ \dots\dots \end{bmatrix}$$

Problem 3 Show that $\text{proj}_{\mathbf{d}} \mathbf{v}$ does not depend on the length of $\mathbf{d}$ by proving that $\text{proj}_{\mathbf{d}} \mathbf{v} = \text{proj}_{k\mathbf{d}} \mathbf{v}$ for $k \neq 0$. What does this result mean geometrically? Illustrate your response with a diagram.

Problem 4 Find the projection of vector $\mathbf{v}$ onto line l . (If entering answers in decimal form, round to the nearest one hundredth.)



Answer:

$$\begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}$$