
1 Span

Linear Combinations Revisited

Recall that a vector $\mathbf{v}$ is said to be a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ if

$$\mathbf{v} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_n \mathbf{v}_n$$

for some scalars $a_1, a_2, \dots, a_n$.

Example 1. If possible, express the given vector as a linear combination of

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

Interpret your results geometrically.

1.

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

2.

$$\mathbf{w} = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$$

Explanation. ?? We need to find coefficients a_1 and a_2 such that $\mathbf{u} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2$. To do this we need to solve the vector equation:

$$a_1 \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

This equation translates into the following system:

$$\begin{aligned} -2a_1 + 2a_2 &= 2 \\ -3a_1 + 3a_2 &= 3 \\ 4a_1 + 2a_2 &= 5 \end{aligned}$$

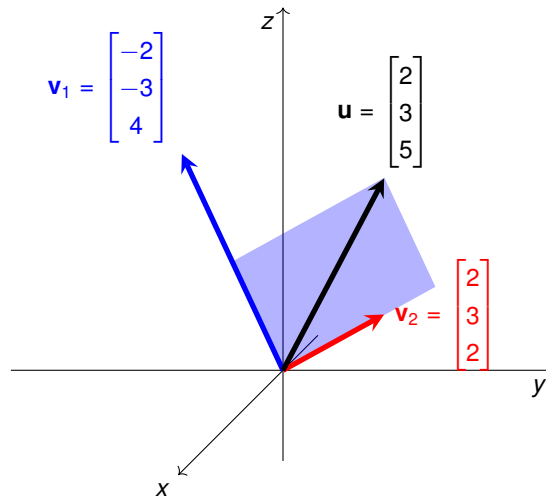
We write the system in augmented matrix form and apply elementary row operations to bring it to reduced row-echelon form.

$$\left[\begin{array}{cc|c} -2 & 2 & 2 \\ -3 & 3 & 3 \\ 4 & 2 & 5 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{array} \right]$$

This shows that $a_1 = \frac{1}{2}$ and $a_2 = \frac{3}{2}$, and we can express $\mathbf{u}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ as follows:

$$\frac{1}{2} \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Observe that because vector $\mathbf{u}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, $\mathbf{u}$ is the diagonal of a parallelogram whose sides are scalar multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$. As such, $\mathbf{u}$ lies in the same plane as $\mathbf{v}_1$ and $\mathbf{v}_2$, as illustrated below.



?? We need to solve the following vector equation:

$$a_1 \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$$

This equation corresponds to the system:

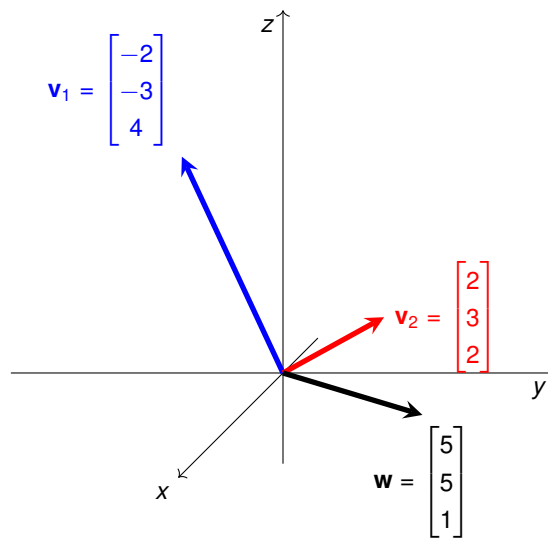
$$\begin{aligned} -2a_1 + 2a_2 &= 5 \\ -3a_1 + 3a_2 &= 5 \\ 4a_1 + 2a_2 &= 1 \end{aligned}$$

Writing the system in augmented matrix form and applying elementary row operations gives us the following reduced row-echelon form:

$$\left[\begin{array}{cc|c} -2 & 2 & 5 \\ -3 & 3 & 5 \\ 4 & 2 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

We conclude that there are no solutions, and $\mathbf{w}$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Geometrically, this means that $\mathbf{w}$ is not the diagonal of any parallelogram whose sides are scalar multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus, $\mathbf{w}$ does not lie in the plane determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.



In part ?? of Example ?? we expressed $\mathbf{u}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, and concluded that $\mathbf{u}$ lies in the plane determined by $\mathbf{v}_1$ and $\mathbf{v}_2$. We say that $\mathbf{u}$ is *in the span* of $\mathbf{v}_1$ and $\mathbf{v}_2$. In fact, every vector in the plane determined by $\mathbf{v}_1$ and $\mathbf{v}_2$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. We say that $\mathbf{v}_1$ and $\mathbf{v}_2$ *span the plane*.

In contrast, vector $\mathbf{w}$ of part ?? of Example ?? is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. We say that $\mathbf{w}$ is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

The following video takes another look at Example ?? using our new vocabulary.

YouTube link: <https://www.youtube.com/watch?v=Sum3qo0EXhY>

Definition of Span

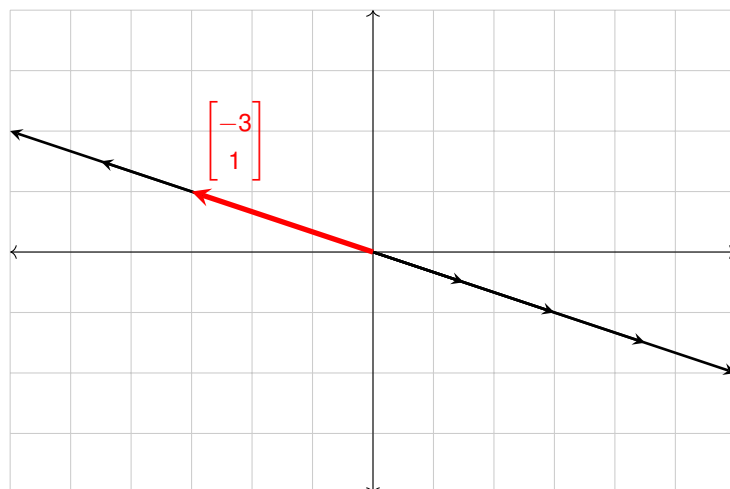
Definition 1. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$. The set S of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is called the *span* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. We write

$$S = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p)$$

and we say that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ *span* S . Any vector in S is said to be *in the span* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is called a *spanning set* for S .

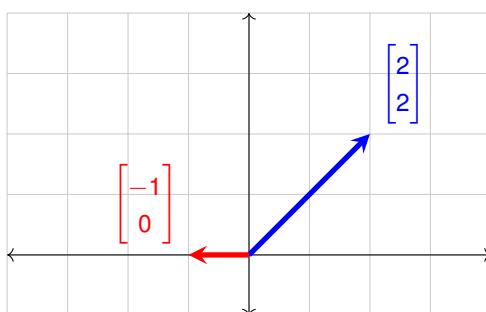
Example 2. Describe $\text{span}\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$.

Explanation. The span of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is the set of all linear combinations of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Since we are looking for linear combinations of only one vector, we are really looking for all of its scalar multiples. So, the span will be the set of all vectors of the form $\mathbf{v} = a \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. All such vectors lie on the line determined by $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.



Example 3. Describe $\text{span} \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$.

Explanation. First, observe that $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ are not scalar multiples of each other.



Geometrically, we can use Procedure ?? to express any vector of $\mathbb{R}^2$ as a linear combination of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, indicating that the two vectors span all of $\mathbb{R}^2$.

To verify this claim algebraically we will show that an arbitrary vector $\begin{bmatrix} s \\ t \end{bmatrix}$ of $\mathbb{R}^2$ can be written as a linear combination of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Consider the vector equation:

$$a_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

This corresponds to the system:

$$\begin{array}{rcl} 2a_1 & - & a_2 = s \\ 2a_1 & & = t \end{array}$$

Writing the system in augmented matrix form and applying elementary row operations gives us the following reduced row-echelon form:

$$\left[\begin{array}{cc|c} 2 & -1 & s \\ 2 & 0 & t \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & t/2 \\ 0 & 1 & t-s \end{array} \right]$$

This shows that every vector of $\mathbb{R}^2$ can be written as a linear combination of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$:

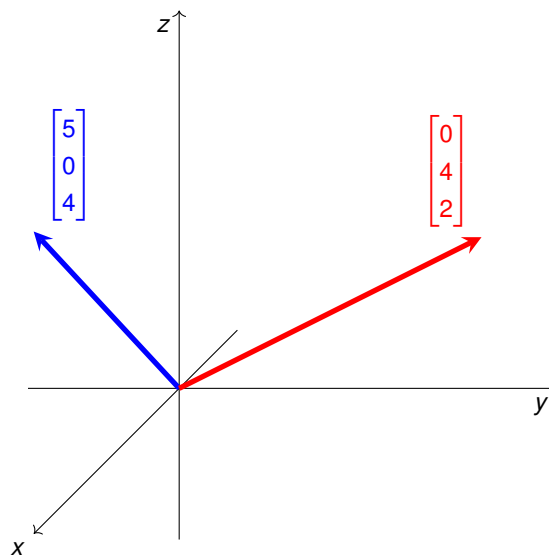
$$(t/2) \begin{bmatrix} 2 \\ 2 \end{bmatrix} + (t-s) \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

We conclude that

$$\text{span} \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) = \mathbb{R}^2$$

Example 4. Describe $\text{span} \left(\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \right)$.

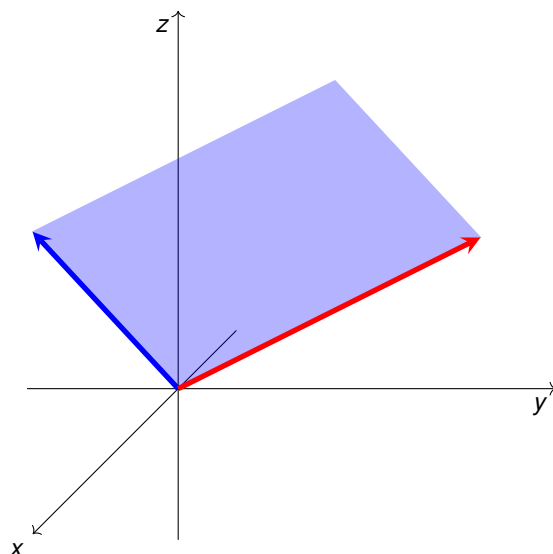
Explanation. First, observe that $\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ are not scalar multiples of each other.



The span of $\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ consists of elements of the form

$$a_1 \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

Geometrically, we can interpret all such linear combinations as diagonals of parallelograms determined by scalar multiples of $\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$. All such diagonals will lie in the plane determined by $\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$. Let this plane be called p . A portion of p is shown below.



Because Procedure ?? can be applied to vectors that lie in p just as easily as it can be applied to vectors of $\mathbb{R}^2$, we conclude that every vector in p can be expressed as a linear combination of $\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$. Thus,

$$\text{span} \left(\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \right) = p$$

Redundant Vectors

If a friend told you that they have a line spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, you would probably think that your friend's description is a little excessive. Isn't one of the above vectors sufficient to describe the line? A line can be

described as a span of one vector, but it can also be described as a span of two or more vectors. There are many advantages, however, to using the most efficient description possible. In this section we will begin to explore what makes a description “more efficient.”

Exploration 1 Consider the following collection of vectors:

$$\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

What is the span of these vectors? (A line, / $\mathbb{R}^2$, / A parallelogram, / A parallelepiped)

In this Exploration we will examine what can happen to the span of a collection of vectors when a vector is removed from the collection.

First, let's remove $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ from $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

Which of the following is true?

Multiple Choice:

1. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$
2. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right)$ is a line
3. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right) = \mathbb{R}^2$
4. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right)$ is a parallelogram.

Removing $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ from $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ (changed, / did not change) the span.

Now let's remove $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ from the original collection of vectors.

Which of the following is true?

Multiple Choice:

1. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$
2. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$ is a line

3. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$ is the right side of the coordinate plane.

4. $\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$ is a parallelogram.

Removing $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ from $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ (changed, / did not change) the span.

As you just discovered, removing a vector from a collection of vectors may or may not affect the span of the collection. We will refer to vectors that can be removed from a collection without changing the span as *redundant*. In Exploration ??, $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ is *redundant*, while $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not.

Definition 2. Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. If we can remove one vector without changing the span of this set, then that vector is *redundant*. In other words, if

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k)$$

we say that $\mathbf{v}_j$ is a redundant element of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, or simply redundant.

Our next goal is to see what causes $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ of Exploration ?? to be redundant. The answer lies not in the vector itself,

but in its relationship to the other vectors in the collection. Observe that $\begin{bmatrix} -4 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. In other words, $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$

is a scalar multiple of another vector in the set. To see why this matters, let's pick an arbitrary vector $\mathbf{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ in

$\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$. Vector $\mathbf{w}$ is in the span because it can be written as a linear combination of the three vectors as follows

$$\mathbf{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

But $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ is not essential to this linear combination because it can be replaced with $-2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, as shown below.

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = - \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Regardless of what vector $\mathbf{w}$ we write as a linear combination of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we will always be able to

replace $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ with $-2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, placing $\mathbf{w}$ into the span of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and making $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ redundant. (Note that

we can just as easily write $\begin{bmatrix} 2 \\ -1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, and argue that $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is redundant.) We conclude that only one of $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is needed to maintain the span of the original three vectors. We have

$$\text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$$

The left-most collection in this expression contains redundant vectors; the other two collections do not.

In Exploration ?? we found one vector to be redundant because we could replace it with a scalar multiple of another vector in the set. The following Exploration delves into what happens when a vector in a given set is a linear combination of the other vectors.

Exploration 2 Consider the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right\}$$

The three vectors are shown below. RIGHT-CLICK and DRAG to rotate the interactive graph.

Access GeoGebra interactives through the online version of this text at

<https://ximera.osu.edu/linearalgebradv3/LinearAlgebraInteractiveIntro>.

$\text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right)$ is (A line, / A plane, / $\mathbb{R}^3$, / A parallelepiped)

Can we remove one of the vectors from the set without changing the span? Observe that we can write $\begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$ as a linear combination of the other two vectors

$$\begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad (1)$$

This means that we can write any vector in $\text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right)$ as a linear combination of only $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and

$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ by replacing $\begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$ with the expression in (??). For example,

$$\begin{bmatrix} 7 \\ 6 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} = \dots \dots \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \dots \dots \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

We have

$$\text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

We conclude that vector $\begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$ is redundant. Can each of the other two vectors in the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right\}$ be considered redundant? You will address this question in the problem set.

Collections of vectors that do not contain redundant vectors are very important in linear algebra. In subsequent sections, we will formally introduce such collections as *linearly independent*. Collections of vectors that contain redundant vectors will be called *linearly dependent*.

Practice Problems

Problem 3 Choose the best description for the set below.

$$\text{span} \left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} \right)$$

Multiple Choice:

1. Plane in $\mathbb{R}^3$
2. Line in $\mathbb{R}^2$
3. Line in $\mathbb{R}^3$
4. $\mathbb{R}^3$
5. $\mathbb{R}^2$

Problem 4 Choose the best description for the set below.

$$\text{span} \left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

Multiple Choice:

1. Plane in $\mathbb{R}^3$
2. Line in $\mathbb{R}^2$
3. Line in $\mathbb{R}^3$

4. $\mathbb{R}^3$

5. $\mathbb{R}^2$

Problem 5 Choose the best description for the set below.

$$\text{span} \left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right)$$

Multiple Choice:

1. Plane in $\mathbb{R}^3$

2. Line in $\mathbb{R}^2$

3. Line in $\mathbb{R}^3$

4. $\mathbb{R}^3$

5. $\mathbb{R}^2$

Problem 6 Which of the following pairs of sets are equal? Select all that apply.

Select All Correct Answers:

1. $V = \text{span} \left(\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix} \right)$ and $W = \text{span} \left(\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

2. $V = \text{span} \left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right)$ and $W = \text{span} \left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

3. $V = \text{span} \left(\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right)$ and $W = \text{span} \left(\begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

4. $V = \text{span} \left(\begin{bmatrix} 5 \\ 0 \end{bmatrix} \right)$ and $W = \text{span} \left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$

Problem 7 Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Give an example of at least one vector $\mathbf{w}$ such that $\mathbf{v}, \mathbf{w}$ do NOT span a plane in $\mathbb{R}^3$.
Describe $\text{span}(\mathbf{v}, \mathbf{w})$.

Problem 8 Prove or disprove. The zero vector of $\mathbb{R}^n$ is contained in the span of any collection of vectors of $\mathbb{R}^n$.

Problem 9 In Exploration ?? we considered the following set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right\}$$

and demonstrated that $\begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$ is redundant by using the fact that it is a linear combination of the other two vectors.

1. Express each of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the remaining vectors.

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \dots\dots \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \dots\dots \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \dots\dots \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \dots\dots \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix}$$

2. Which of the following is NOT true?

Multiple Choice:

- (i) If $\mathbf{w}$ is in $\text{span} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right)$, then $\mathbf{w}$ is in $\text{span} \left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right)$.
- (ii) Both $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ are redundant in $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right\}$.
- (iii) We can remove $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ from $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \end{bmatrix} \right\}$ at the same time without affecting the span.

Problem 10 Show that if the zero vector is part of a collection of two or more vectors, the zero vector is redundant.