
1 Essential Vocabulary

Coordinate vector

Let V be a vector space, and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V . If $\mathbf{v} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$, then the vector in $\mathbb{R}^n$ whose components are the coefficients $a_1, \dots, a_n$ is said to be the *coordinate vector for $\mathbf{v}$ with respect to $\mathcal{B}$* . We denote the coordinate vector by $[\mathbf{v}]_{\mathcal{B}}$ and write:

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Finite-dimensional vector space

A vector space is said to be *finite-dimensional* if it is spanned by finitely many vectors.

Isomorphism

Let V and W be vector spaces. If there exists an invertible linear transformation $T : V \rightarrow W$ we say that V and W are *isomorphic* and write $V \cong W$. The invertible linear transformation T is called an *isomorphism*.

Matrix of a linear transformation

Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\mathcal{C}$, respectively. Suppose $T : V \rightarrow W$ is a linear transformation.

$$\text{Let } A = \begin{bmatrix} | & | & & | \\ [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \dots & [T(\mathbf{v}_n)]_{\mathcal{C}} \\ | & | & & | \end{bmatrix}$$

Then $A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$ for all vectors $\mathbf{v}$ in V .

Matrix A is called the matrix of T with respect to ordered bases $\mathcal{B}$ and $\mathcal{C}$.

One-to-one linear transformation

A linear transformation $T : V \rightarrow W$ is *one-to-one* if

$$T(\mathbf{v}_1) = T(\mathbf{v}_2) \text{ implies that } \mathbf{v}_1 = \mathbf{v}_2$$

Onto linear transformation

A linear transformation $T : V \rightarrow W$ is *onto* if for every element $\mathbf{w}$ of W , there exists an element $\mathbf{v}$ of V such that $T(\mathbf{v}) = \mathbf{w}$.

Subspace

A nonempty subset U of a vector space V is called a *subspace* of V , provided that U is itself a vector space when given the same addition and scalar multiplication as V .

Vector Space

Let V be a nonempty set. Suppose that elements of V can be added together and multiplied by scalars. The set V , together with operations of addition and scalar multiplication, is called a *vector space* provided that

V is closed under addition

V is closed under scalar multiplication

and the following properties hold for $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and scalars k and p :

1. Commutative Property of Addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. Associative Property of Addition: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. Existence of Additive Identity: $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. Existence of Additive Inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. Distributive Property over Vector Addition: $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
6. Distributive Property over Scalar Addition: $(k + p)\mathbf{u} = k\mathbf{u} + p\mathbf{u}$
7. Associative Property for Scalar Multiplication: $k(p\mathbf{u}) = (kp)\mathbf{u}$
8. Multiplication by 1: $1\mathbf{u} = \mathbf{u}$

We will refer to elements of V as *vectors*.

Note that definitions of span, linear independence, basis, and dimension are analogous to those for subspaces of $\mathbb{R}^n$.