
1 Essential Vocabulary

Fundamental subspaces of a matrix $\text{null}(A)$ is the orthogonal complement of $\text{row}(A)$, and $\text{null}(A^T)$ is the orthogonal complement of $\text{col}(A)$.

Gram-Schmidt process An iterative process which constructs an orthogonal basis for a subspace. The idea is to build the orthogonal set one vector at a time, by taking a vector not in the span of the vectors in the current iteration of the set, and subtracting its orthogonal projection onto each of those vectors.

Orthogonal Basis A set of orthogonal vectors that spans a subspace. (Any orthogonal set of vectors must be linearly independent.)

Orthogonal complement of a subspace If W is a subspace, we define the *orthogonal complement* $W^\perp$ as the set of all vectors orthogonal to every vector in W , i.e.,

$$W^\perp = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{y} \in W\}$$

Orthogonal Decomposition Theorem Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{x} \in \mathbb{R}^n$. Then there exist unique vectors $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$ such that $\mathbf{x} = \mathbf{w} + \mathbf{w}^\perp$.

Orthogonal matrices An $n \times n$ matrix Q is called an *orthogonal matrix* if its columns form an orthonormal set. This will happen if and only if its rows form an orthonormal set. Note also that Q is an orthogonal matrix if and only if it is an invertible matrix such that $Q^{-1} = Q^T$.

Orthogonal projection onto a subspace Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$. If $\mathbf{x}$ is in $\mathbb{R}^n$, the vector

$$\mathbf{w} = \text{proj}_W(\mathbf{x}) = \text{proj}_{\mathbf{f}_1} \mathbf{x} + \text{proj}_{\mathbf{f}_2} \mathbf{x} + \dots + \text{proj}_{\mathbf{f}_m} \mathbf{x}$$

is called the *orthogonal projection* of $\mathbf{x}$ onto W .

Orthogonal set of vectors Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of nonzero vectors in $\mathbb{R}^n$. Then this set is called an *orthogonal set* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$. Moreover, if $\|\mathbf{v}_i\| = 1$ for $i = 1, \dots, m$ (i.e. each vector in the set is a unit vector), we say the set of vectors is an *orthonormal set*.

Orthogonally diagonalizable matrix An $n \times n$ matrix A is said to be *orthogonally diagonalizable* if an orthogonal matrix Q can be found such that $Q^{-1}AQ = Q^T A Q$ is diagonal.

Orthonormal basis A set of orthonormal vectors that spans a subspace. (Any orthogonal set of vectors must be linearly independent by Theorem ??.)

Orthonormal set of vectors Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of nonzero vectors in $\mathbb{R}^n$. Then this set is called an *orthogonal set* if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$. Moreover, if $\|\mathbf{v}_i\| = 1$ for $i = 1, \dots, m$ (i.e. each vector in the set is a unit vector), we say the set of vectors is an *orthonormal set*.

Properties of orthogonal matrices If Q is an orthogonal matrix, then...

1. $Q^{-1} = Q^T$ is orthogonal,
2. $\det Q = \pm 1$,
3. if λ is an eigenvalue of Q , then $|\lambda| = 1$,
4. the product of Q with any other orthogonal matrix will be an orthogonal matrix (i.e. orthogonal matrices are closed under matrix multiplication),
and
5. Q is a length-preserving and angle-preserving linear transformation.

QR factorization Let A be an $m \times n$ matrix with independent columns. A *QR-factorization* of A expresses it as $A = QR$ where Q is $m \times n$ with orthonormal columns and R is an invertible and upper triangular matrix with positive diagonal entries.

Rank-Nullity Theorem for matrices (??) Let A be an $m \times n$ matrix. Then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

This implies that the dimension of a subspace of $\mathbb{R}^n$ plus the dimension of its orthogonal complement equals n .

Spectral Theorem If A is a real $n \times n$ matrix, then A is symmetric if and only if A is orthogonally diagonalizable.
