
1 Abstract Vector Spaces

Earlier in the text we discussed $\mathbb{R}^n$ as a vector space and introduced the notion of a subspace of $\mathbb{R}^n$. In this section we will consider sets other than $\mathbb{R}^n$ that have two operations and satisfy the same properties as $\mathbb{R}^n$. Such sets, together with the operations of addition and scalar multiplication, will also be called vector spaces.

Properties of Vector Spaces

Recall that $\mathbb{R}^n$ is said to be a vector space because

$\mathbb{R}^n$ is closed under vector addition

$\mathbb{R}^n$ is closed under scalar multiplication

and satisfies the following properties:

1. Commutative Property of Addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. Associative Property of Addition: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. Existence of Additive Identity: $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. Existence of Additive Inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. Distributive Property over Vector Addition: $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
6. Distributive Property over Scalar Addition: $(k + p)\mathbf{u} = k\mathbf{u} + p\mathbf{u}$
7. Associative Property for Scalar Multiplication: $k(p\mathbf{u}) = (kp)\mathbf{u}$
8. Multiplication by 1: $1\mathbf{u} = \mathbf{u}$

In the next two examples we will explore two sets other than $\mathbb{R}^n$ endowed with addition and scalar multiplication and satisfying the same properties.

Example 1. Let $\mathbb{M}_{m,n}$ be the set of all $m \times n$ matrices. Matrix addition and scalar multiplication were defined in [Addition and Scalar Multiplication of Matrices](#).

Observe that the sum of two $m \times n$ matrices is also an $m \times n$ matrix. Likewise, a scalar multiple of an $m \times n$ matrix is an $m \times n$ matrix. Thus

$\mathbb{M}_{m,n}$ is closed under matrix addition

$\mathbb{M}_{m,n}$ is closed under scalar multiplication

In addition, Theorems ?? and ?? give us the following properties of matrix addition and scalar multiplication. Note that these properties are analogous to the eight vector properties above.

1. Commutative Property of Addition: $A + B = B + A$

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2. Associative Property of Addition: $(A + B) + C = A + (B + C)$
 3. Existence of Additive Identity: $A + O = A$ where O is the $m \times n$ zero matrix
 4. Existence of Additive Inverse: $A + (-A) = O$
 5. Distributive Property over Matrix Addition: $k(A + B) = kA + kB$
 6. Distributive Property over Scalar Addition: $(k + p)A = kA + pA$
 7. Associative Property for Scalar Multiplication: $k(pA) = (kp)A$
 8. Multiplication by 1: $1A = A$

Example 2. Consider the set $\mathbb{L}$ of all linear functions. This set includes all polynomials of degree 1 and degree 0. We will use addition and scalar multiplication of polynomials as the two operations, and show that $\mathbb{L}$ is closed under those operations and satisfies eight properties analogous to those of vectors of $\mathbb{R}^n$.

Explanation. Elements of $\mathbb{L}$ are functions f given by

$$f(x) = mx + b$$

(Note that m and b can be equal to zero.)

Given f_1 and f_2 in $\mathbb{L}$, it is easy to verify that $f_1 + f_2$ is also in $\mathbb{L}$. This gives us closure under function addition.

For any scalar k , we have

$$kf(x) = k(mx + b) = (km)x + (kb)$$

Therefore kf is in $\mathbb{L}$, and $\mathbb{L}$ is closed under scalar multiplication.

We now proceed to formulate eight properties analogous to those of vectors of $\mathbb{R}^n$.

Let f_1 , f_2 and f_3 be elements of $\mathbb{L}$ given by $f_1(x) = m_1x + b_1$, $f_2(x) = m_2x + b_2$, and $f_3(x) = m_3x + b_3$. Let k and p be scalars.

1. Commutative Property of Addition: $f_1 + f_2 = f_2 + f_1$

This property holds because

$$\begin{aligned} f_1(x) + f_2(x) &= (m_1x + b_1) + (m_2x + b_2) \\ &= (m_2x + b_2) + (m_1x + b_1) \\ &= f_2(x) + f_1(x) \end{aligned}$$

2. Associative Property of Addition: $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$

This property is easy to verify.

3. Existence of Additive Identity: $f_1 + f_0 = f_1$

The additive identity f_0 is given by $f_0(x) = 0$. Note that f_0 is in $\mathbb{L}$.

4. Existence of Additive Inverse: $f_1 + (-f_1) = f_0$

The additive inverse of f_1 is a function $-f_1$ given by $-f_1(x) = -mx + (-b)$. Note that $-f_1$ is in $\mathbb{L}$.

5. Distributive Property over Vector Addition: $k(f_1 + f_2) = kf_1 + kf_2$

This property holds because

$$\begin{aligned}k(f_1(x) + f_2(x)) &= k((m_1x + b_1) + (m_2x + b_2)) \\&= k(m_1x + b_1) + k(m_2x + b_2) \\&= kf_1(x) + kf_2(x)\end{aligned}$$

6. Distributive Property over Scalar Addition: $(k + p)f_1 = kf_1 + pf_1$

This property holds because

$$\begin{aligned}(k + p)f_1(x) &= (k + p)(m_1x + b_1) \\&= k(m_1x + b_1) + p(m_1x + b_1) \\&= kf_1(x) + pf_1(x)\end{aligned}$$

7. Associative Property for Scalar Multiplication: $(k(pf_1)) = (kp)f_1$

This property holds because

$$\begin{aligned}k(p(f_1(x))) &= k(p(m_1x + b_1)) \\&= k(pm_1x + pb_1) \\&= kpm_1x + kpb_1 \\&= (kp)m_1x + (kp)b_1 \\&= (kp)(m_1x + b_1) \\&= (kp)f_1(x)\end{aligned}$$

8. Multiplication by 1

$$1f_1 = f_1$$

Definition of a Vector Space

Examples ?? and ?? show us that there are many times in mathematics when we encounter a set with two operations (that we call addition and scalar multiplication) such that the set is closed under the two operations, and satisfies the same eight properties as $\mathbb{R}^n$. We will refer to such sets as *vector spaces*.

Definition 1. Let V be a nonempty set. Suppose that elements of V can be added together and multiplied by scalars. The set V , together with operations of addition and scalar multiplication, is called a *vector space* provided that

V is closed under addition

V is closed under scalar multiplication

and the following properties hold for $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and scalars k and p :

1. Commutative Property of Addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

2. Associative Property of Addition: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

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3. Existence of Additive Identity: $\mathbf{u} + \mathbf{0} = \mathbf{u}$
 4. Existence of Additive Inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
 5. Distributive Property over Vector Addition: $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
 6. Distributive Property over Scalar Addition: $(k + p)\mathbf{u} = k\mathbf{u} + p\mathbf{u}$
 7. Associative Property for Scalar Multiplication: $k(p\mathbf{u}) = (kp)\mathbf{u}$
 8. Multiplication by 1: $1\mathbf{u} = \mathbf{u}$

We will refer to elements of V as *vectors*.

When scalars k and p in the above definition are restricted to real numbers, as they are in this chapter, vector space V may be referred to as a *vector space over the real numbers*.

Example 3. $\mathbb{M}_{m,n}$ and $\mathbb{L}$ are vector spaces. (See Examples ?? and ??)

Sets of polynomials provide an important source of examples, so we review some basic facts. A *polynomial with real coefficients* in x is an expression

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

where $a_0, a_1, a_2, \dots, a_n$ are real numbers called the *coefficients* of the polynomial. If all the coefficients are zero, the polynomial is called the *zero polynomial* and is denoted simply as 0. If $p(x) \neq 0$, the highest power of x with a nonzero coefficient is called the *degree* of $p(x)$ denoted as $\deg(p(x))$. The coefficient itself is called the *leading coefficient* of $p(x)$. Hence $\deg(3 + 5x) = 1$, $\deg(1 + x + x^2) = 2$, and $\deg(4) = 0$. (The degree of the zero polynomial is not defined.)

Let $\mathbb{P}$ denote the set of all polynomials and suppose that

$$\begin{aligned} p(x) &= a_0 + a_1x + a_2x^2 + \dots \\ q(x) &= b_0 + b_1x + b_2x^2 + \dots \end{aligned}$$

are two polynomials in $\mathbb{P}$ (possibly of different degrees). Then $p(x)$ and $q(x)$ are called *equal* (written $p(x) = q(x)$) if and only if all the corresponding coefficients are equal—that is, $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, and so on. In particular, $a_0 + a_1x + a_2x^2 + \dots = 0$ means that $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, ...

The set $\mathbb{P}$ has an addition and scalar multiplication defined on it as follows: if $p(x)$ and $q(x)$ are as before and k is a real number,

$$\begin{aligned} p(x) + q(x) &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots \\ kp(x) &= ka_0 + (ka_1)x + (ka_2)x^2 + \dots \end{aligned}$$

Example 4. $\mathbb{P}$ is a vector space.

Explanation. It is easy to see that the sum of two polynomials is again a polynomial, and that a scalar multiple of a polynomial is a polynomial. Thus, $\mathbb{P}$ is closed under addition and scalar multiplication. The other eight vector space properties are easily verified, and we conclude that $\mathbb{P}$ is a vector space.

Example 5. Let Y be the set of all degree two polynomials in x . In other words,

$$Y = \{ax^2 + bx + c : a \neq 0\}$$

We claim that Y is not a vector space.

Explanation. Observe that Y is not closed under addition. To see this, let $y_1 = 2x^2 + 3x + 4$ and let $y_2 = -2x^2$. Then y_1 and y_2 are both elements of Y . However, $y_1 + y_2 = 3x + 4$ is not an element of Y , as it is only a degree one polynomial. We require the coefficient a of x^2 to be nonzero for a polynomial to be in Y , and this is not the case for $y_1 + y_2$.

As an exercise, check the remaining vector space properties one-by-one to see which properties hold and which do not.

Set Y in Example ?? is not a vector space, but if we make a slight modification, we can make it into a vector space.

Example 6. Let $\mathbb{P}^2$ be the set of polynomials of degree two or less. In other words,

$$\mathbb{P}^2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$$

Note that $\mathbb{P}^2$ contains the zero polynomial (let $a = b = c = 0$).

Unlike set Y in Example ??, $\mathbb{P}^2$ is closed under polynomial addition and scalar multiplication. It is easy to verify that all vector space properties hold, so $\mathbb{P}^2$ is a vector space.

Example 7. Let n be a natural number. Define $\mathbb{P}^n$ to be the set of polynomials of degree n or less than n , then by reasoning similar to Example ??, $\mathbb{P}^n$ is a vector space.

Subspaces

Definition 2. A nonempty subset U of a vector space V is called a *subspace* of V , provided that U is itself a vector space when given the same addition and scalar multiplication as V .

Example 8. In Example ?? we demonstrated that $\mathbb{P}^2$ is a vector space. From Example ?? we know that $\mathbb{P}$ is a vector space. But $\mathbb{P}^2$ is a subset of $\mathbb{P}$, and uses the same operations of polynomial addition and scalar multiplication. Therefore $\mathbb{P}^2$ is a subspace of $\mathbb{P}$.

Checking all ten properties to verify that a subset of a vector space is a subspace can be cumbersome. Fortunately we have the following theorem.

Theorem 1 (Subspace Test). Let U be a nonempty subset of a vector space V . If U is closed under the operations of addition and scalar multiplication of V , then U is a subspace of V .

Proof To prove that closure is a sufficient condition for U to be a subspace, we will need to show that closure under addition and scalar multiplication of V guarantees that the remaining eight properties are satisfied automatically.

Observe that Properties ??, ??, ??, ??, ?? and ?? hold for all elements of V . Thus, these properties will hold for all elements of U . We say that these properties are *inherited* from V .

To prove Property ?? we need to show that $\mathbf{0}$, which we know to be an element of V , is contained in U . Let $\mathbf{u}$ be an element of U (recall that U is nonempty). We will show that $0\mathbf{u} = \mathbf{0}$ in V . Then, by closure under scalar multiplication, we will be able to conclude that $0\mathbf{u} = \mathbf{0}$ must be in U .

$$0\mathbf{u} = (0 + 0)\mathbf{u} = 0\mathbf{u} + 0\mathbf{u}$$

Adding the additive inverse of $0\mathbf{u}$ to both sides gives us

$$0\mathbf{u} + (-0\mathbf{u}) = (0\mathbf{u} + 0\mathbf{u}) + (-0\mathbf{u})$$

By Properties ?? and ??

$$\mathbf{0} = 0\mathbf{u} + (0\mathbf{u} + (-0\mathbf{u}))$$

By Properties ?? and ??

$$\mathbf{0} = 0\mathbf{u} + \mathbf{0} = 0\mathbf{u}$$

Because U is closed under scalar multiplication $0\mathbf{u} = \mathbf{0}$ is in U .

We know that every element of U , being an element of V , has an additive inverse in V . We need to show that the additive inverse of every element of U is contained in U . Let $\mathbf{u}$ be any element of U . We will show that $(-1)\mathbf{u}$ is the additive inverse of $\mathbf{u}$. Then by closure, $(-1)\mathbf{u}$ will have to be contained in U . To show that $(-1)\mathbf{u}$ is the additive inverse of $\mathbf{u}$, we must show that $\mathbf{u} + (-1)\mathbf{u} = \mathbf{0}$. We compute:

$$\mathbf{u} + (-1)\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u} = (1 + (-1))\mathbf{u} = 0\mathbf{u} = \mathbf{0}$$

Thus $(-1)\mathbf{u}$ is the additive inverse of $\mathbf{u}$. By closure, $(-1)\mathbf{u}$ is in U . ■

Example 9. Let A be a fixed matrix in $\mathbb{M}_{n,n}$. Show that the set C_A of all $n \times n$ matrices that commute with A under matrix multiplication is a subspace of $\mathbb{M}_{n,n}$.

Explanation. The set C_A consists of all $n \times n$ matrices X such that $AX = XA$. First, observe that C_A is not empty because I_n is an element. Now we need to show that C_A is closed under matrix addition and scalar multiplication.

Suppose that X_1 and X_2 lie in C_A . Then $AX_1 = X_1A$ and $AX_2 = X_2A$. Then

$$A(X_1 + X_2) = AX_1 + AX_2 = X_1A + X_2A = (X_1 + X_2)A$$

Therefore $(X_1 + X_2)$ commutes with A . Thus $(X_1 + X_2)$ is in C_A . We conclude that C_A is closed under matrix addition.

Now suppose X is in C_A . Let k be a scalar, then

$$A(kX) = k(AX) = k(XA) = (kX)A$$

Therefore (kX) commutes with A . We conclude that (kX) is in C_A , and C_A is closed under scalar multiplication. Hence C_A is a subspace of $\mathbb{M}_{n,n}$.

Suppose $p(x)$ is a polynomial and a is a number. Then the number $p(a)$ obtained by replacing x by a in the expression for $p(x)$ is called the *evaluation* of $p(x)$ at a . For example, if $p(x) = 5 - 6x + 2x^2$, then the evaluation of $p(x)$ at $a = 2$ is $p(2) = 5 - 12 + 8 = 1$. If $p(a) = 0$, the number a is called a *root* of $p(x)$.

Example 10. Consider the set U of all polynomials in $\mathbb{P}$ that have 3 as a root:

$$U = \{p(x) \in \mathbb{P} : p(3) = 0\}$$

Show that U is a subspace of $\mathbb{P}$.

Explanation. Observe that U is not empty because $r(x) = x - 3$ is an element of U . Suppose $p(x)$ and $q(x)$ lie in U . Then $p(3) = 0$ and $q(3) = 0$. We have $(p + q)(x) = p(x) + q(x)$ for all x , so $(p + q)(3) = p(3) + q(3) = 0 + 0 = 0$, and U is closed under addition. The verification that U is closed under scalar multiplication is similar.

Linear Combinations and Span

Definition 3. Let V be a vector space and let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be vectors in V . A vector $\mathbf{v}$ is said to be a *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ if

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n$$

for some scalars $a_1, a_2, \dots, a_n$.

Definition 4. Let V be a vector space and let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ be vectors in V . The set S of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is called the *span* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. We write

$$S = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p)$$

and we say that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ *span* S . Any vector in S is said to be *in the span* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is called a *spanning set* for S .

Example 11. Consider $p_1 = 1 + x + 4x^2$ and $p_2 = 1 + 5x + x^2$ in $\mathbb{P}^2$. Determine whether p_1 and p_2 lie in $\text{span}\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Explanation. For p_1 , we want to determine if a and b exist such that

$$p_1 = a(1 + 2x - x^2) + b(3 + 5x + 2x^2)$$

Expanding the right hand side gives us:

$$a + 2ax - ax^2 + 3b + 5bx + 2bx^2$$

Combining like terms, we get:

$$(a + 3b) + (2a + 5b)x + (-a + 2b)x^2$$

Setting this equal to $p_1 = 1 + x + 4x^2$ and equating coefficients of powers of x gives us a system of equations

$$1 = a + 3b, \quad 1 = 2a + 5b, \quad \text{and} \quad 4 = -a + 2b$$

This system has the solution $a = -2$ and $b = 1$, so p_1 is indeed in $\text{span}\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Turning to $p_2 = 1 + 5x + x^2$, we are looking for a and b such that

$$p_2 = a(1 + 2x - x^2) + b(3 + 5x + 2x^2)$$

Again equating coefficients of powers of x gives equations $1 = a + 3b$, $5 = 2a + 5b$, and $1 = -a + 2b$. But in this case there is no solution, so p_2 is not in $\text{span}\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Theorem 2. Let V be a vector space. Let S be any subset of V . Then $U = \text{span}(S)$ is a subspace of V .

Proof See Practice Problem ??.



Practice Problems

Problem 1 Given the set of all points in $\mathbb{R}^2$, define the operations of addition and scalar multiplication as follows:

Addition: $(a, b) + (c, d) = (a + d, b + c)$

Scalar Multiplication: $k(a, b) = (ka, kb)$

Is addition, as defined above, commutative?

Multiple Choice:

1. Yes
2. No

Problem 2 Given the set of all points in $\mathbb{R}^2$, define the operations of addition and scalar multiplication as follows:

Addition: $(a, b) + (c, d) = (0, b + d)$

Scalar Multiplication: $k(a, b) = (ka, kb)$

Is addition, as defined above, commutative?

Multiple Choice:

1. Yes
2. No

Is there an additive identity?

Multiple Choice:

1. Yes
2. No

Problem 3 Given the set of all points in $\mathbb{R}^2$, define the operations of addition and scalar multiplication as follows:

Addition: $(a, b) + (c, d) = (a + c, b + d)$

Scalar Multiplication: $k(a, b) = (a, kb)$

Is the Distributive Property over Vector Addition satisfied?

Multiple Choice:

1. Yes
2. No

Is the Distributive Property over Scalar Addition satisfied?

Multiple Choice:

1. Yes
2. No

Problem 4 Let $\mathcal{F}$ be the set of all real-valued functions whose domain is all real numbers. Define addition and scalar multiplication as follows:

$$(f + g)(x) = f(x) + g(x) \quad (cf)(x) = cf(x)$$

Verify that $\mathcal{F}$ is a vector space.

Problem 5 Recall that for a fixed matrix A in $\mathbb{M}_{n,n}$, the set C_A of all $n \times n$ matrices that commute with A under matrix multiplication is a subspace of $\mathbb{M}_{n,n}$. (Example ??)

Describe all elements of C_I , where I is a 3×3 identity matrix.

Problem 6 Is the subset of all invertible $n \times n$ matrices a subspace of $\mathbb{M}_{n,n}$?

Multiple Choice:

1. Yes
2. No

Prove your claim.

Problem 7 Is the subset of all symmetric $n \times n$ matrices a subspace of $\mathbb{M}_{n,n}$?

Multiple Choice:

1. Yes
2. No

Prove your claim.

Problem 8 Let Z be a subset of $\mathbb{M}_{n,n}$ that consists of $n \times n$ matrices that commute with every matrix in $\mathbb{M}_{n,n}$ under matrix multiplication. In other words,

$$Z = \{B : BY = YB \text{ for all } Y \in \mathbb{M}_{n,n}\}$$

Is Z a subspace of $\mathbb{M}_{n,n}$?

Problem 9 List several elements of $\text{span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$. Suggest a spanning set for $\mathbb{M}_{2,2}$.

Problem 10 Find $\text{span}(1, x, x^2, x^3)$.

Problem 11 Let V be a vector space. Let S be any subset of V . Prove that $U = \text{span}(S)$ is a subspace of V . (Theorem ??)

Text Source

The discussion on polynomials was adapted from Section 6.1 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p. 331-332

Example Source

Examples ?? and ?? were adapted from Examples 6.2.4 and 6.2.7 of Keith Nicholson's *Linear Algebra with Applications*. (CC-BY-NC-SA)

W. Keith Nicholson, *Linear Algebra with Applications*, Lyryx 2018, Open Edition, p. 340-341

Exercise Source

Practice Problems ??-?? is adopted from Problems 9.1.1-9.1.4 of Ken Kuttler's *A First Course in Linear Algebra*. (CC-BY)

Ken Kuttler, *A First Course in Linear Algebra*, Lyryx 2017, Open Edition, p. 469.